Cross-Return, Showrooming, and Online-Offline Competition

Online Appendix

A. Consumer Segmentations

We first derive their shopping choices by comparing U_S, U_F, and U_E. We find that $U_E > U_S$ when $h_O < \hat{h}^i_{OES} = 2l - \phi$. For no-cross-return case, we further derive $\hat{h}^N_{OES} = 2l - h_r$. We consider that $l > h_r / 2$, such that showrooming will not dominate e-Direct. For cross-return case, we get $\hat{h}^C_{OES} = l$. We find $U_S > U_F$ when $h_O < \hat{h}^i_{OSF} = p_F - p_O$ for both cross- and no-cross-return cases.

Then we separate our analysis into two cases: (i) $\hat{h}^i_{OES} \le \hat{h}^i_{OSF}$ and (ii) $\hat{h}^i_{OES} > \hat{h}^i_{OSF}$. For the case with $\hat{h}^i_{OES} \le \hat{h}^i_{OSF}$, we get $p_O \le \hat{p}^i_{O2} = p_F - 2l + \phi$, which indicates $\hat{p}^N_{O2} = p_F - 2l + h_r$ and $\hat{p}^C_{O2} = p_F - l$. Then, we find that (i) $U_E > \max\{U_S, U_F\}$ for $0 \le h_O < \hat{h}^i_{OES}$, and (ii) $U_S \ge \max\{U_E, U_F\}$ for $\hat{h}^i_{OES} \le h_O \le \hat{h}^i_{OSF}$. If we further have $\hat{h}^i_{OSF} \le 1$, i.e., $p_O \ge \hat{p}^i_{O3} = p_F - 1$, we will have $U_F > \max\{U_S, U_E\}$ for $\hat{h}^i_{OSF} < h_O \le 1$. To summarize, when $\hat{p}^i_{O3} < p_O \le \hat{p}^i_{O2}$, the consumers with $0 \le h_O < \hat{h}^i_{OES}$ will choose e-Direct, the consumers with $\hat{h}^i_{OES} \le h_O \le \hat{h}^i_{OSF}$ will choose showrooming, and the consumers with $\hat{h}^i_{OSF} < h_O \le 1$ will choose buy-offline. If $\hat{h}^i_{OSF} > 1$, i.e., $p_O < \hat{p}^i_{O3}$, none of the consumers will choose buy-offline. The consumers with $0 \le h_O < \hat{h}^i_{OES}$ will choose e-Direct, and the consumers with $\hat{h}^i_{OES} \le h_O \le 1$ will choose showrooming. We assume that $l < (1 + h_r)/2$ in order to have $\hat{p}^N_{O3} < \hat{p}^N_{O2}$, otherwise buy-offline and showrooming would not co-exist at any given p_O for no-cross-return case.

For the case with $\hat{h}^i_{OES} > \hat{h}^i_{OSF}$, which indicates $p_O > \hat{p}^i_{O2}$, there does not exist a region for $U_S \ge \max\{U_E, U_F\}$ as it requires $\hat{h}^i_{OES} \le h_O \le \hat{h}^i_{OSF}$. Hence, there is no showrooming consumer in this case. Instead, we find that $U_E > U_F$ when $h_O < \hat{h}^i_{OEF} = (p_F - p_O + 2l - \phi)/2$, which indicates $\hat{h}^N_{OEF} = (p_F - p_O + 2l - h_r)/2$ and $\hat{h}^C_{OEF} = (p_F - p_O + l)/2$. To make sure $\hat{h}^i_{OEF} > 0$, we need $p_O < \hat{p}^i_{O1} = p_F + 2l - \phi$, more specifically, $\hat{p}^N_{O1} = p_F + 2l - h_r$ and $\hat{p}^C_{O1} = p_F + l$. It's trivial to show $\hat{p}^C_{O1} > \hat{p}^C_{O2}$. We can further verify that $\hat{p}^N_{O1} > \hat{p}^N_{O2}$ based on the assumption $h_r / 2 < l < (1 + h_r)/2$. In addition, we find that $0 < \hat{h}^i_{OEF} < 1$ when $\hat{p}^i_{O2} < p_O \le \hat{p}^i_{O1}$. Hence, when $\hat{p}^i_{O2} < p_O \le \hat{p}^i_{O1}$, the consumers with

$0 \le h_O \le \hat{h}^i_{OEF}$ will choose e-Direct, and the consumers with $\hat{h}^i_{OEF} < h_O \le 1$ will choose buy-offline. When $p_O > \hat{p}^i_{O1}$, we have $\hat{h}^i_{OEF} \le 0$. In such a case, the consumers with $0 \le h_O \le 1$ will choose buy-offline.

B. Best-Responses and Profits

We first set up the consumer demand a, based on consumer segmentation from Lemma 1. For simplicity, we introduce the following notation: we use case A to denote Seg F (segment F) from Lemma 1, case B for Seg E-F, case C for Seg E-S-F, and case D for Seg E-S.

- Case A: When $p_O > \hat{p}^i_{O1}$, $a^i_{EA} = 0$, $a^i_{SA} = 0$, $a^i_{FA} = 1/2$;
- Case B: When $\hat{p}^i_{O2} < p_O \le \hat{p}^i_{O1}$, $a^i_{EB} = \hat{h}^i_{OEF}/2$, $a^i_{SB} = 0$, $a^i_{FB} = \left(1-\hat{h}^i_{OEF}\right)/2$;
- Case C: When $\hat{p}^i_{O3} < p_O \le \hat{p}^i_{O2}$, $a^i_{EC} = \hat{h}^i_{OES}/2$, $a^i_{SC} = \left(\hat{h}^i_{OSF} - \hat{h}^i_{OES}\right)/2$, $a^i_{FC} = \left(1-\hat{h}^i_{OSF}\right)/2$;
- Case D: When $p_O \le \hat{p}^i_{O3}$, $a^i_{ED} = \hat{h}^i_{OES}/2$, $a^i_{SD} = \left(1-\hat{h}^i_{OES}\right)/2$, $a^i_{FD} = 0$.

Now let's derive offline retailer's best response functions under each case.

- Case A: When $p_O > \hat{p}^C_{O1}$, we get $p_F < p_O - l$, the total profit function is $\pi_{FA} = \left(p_F\right)\cdot a^C_{FA} + \left(f - s_F\right)\cdot a^C_{EA} = p_F/2$. We derive positive derivative $\frac{d\pi_{FA}}{dp_F} = \frac{1}{2}$, so the best response price for physical retailer is $p^*_F = \hat{p}^C_{F5} = p_{\mathrm{O}} - l$. Thus, the total profit for offline retailer in this case is $\pi^*_{FA} = \frac{p_O - l}{2}$;
- Case B: When $\hat{p}^C_{O2} < p_O \le \hat{p}^C_{O1}$, we get $p_O - l \le p_F < p_O + l$, the total profit function is $\pi_{FB} = \left(p_F\right)\cdot a^C_{FB} + \left(f - s_F\right)\cdot a^C_{EB} = p_F\left(\frac{1}{2} - \frac{l}{4} - \frac{p_F}{4} + \frac{p_{\mathrm{O}}}{4}\right) + \left(f - s_F\right)\left(\frac{l}{4} + \frac{p_F}{4} - \frac{p_{\mathrm{O}}}{4}\right)$. We solve the derivative $\frac{d\pi_{FB}}{dp_F} = 0$ and get $p^*_F = \hat{p}^C_{F4} = \left(p_O + f - s_F - l + 2\right)/2$ and

$$\pi^*_{FB} = -\frac{1}{4}l + \frac{1}{4}p_{\mathrm{O}} + \frac{1}{4} + \frac{1}{16}f^2 - \frac{1}{8}fs_F + \frac{1}{16}s_F^2 - \frac{1}{8}p_{\mathrm{O}}l + \frac{1}{16}p_{\mathrm{O}}^2 + \frac{1}{16}l^2 + \frac{1}{8}lf + \frac{1}{4}f - \frac{1}{8}p_{\mathrm{O}}f - \frac{1}{8}ls_F - \frac{1}{4}s_F + \frac{1}{8}p_{\mathrm{O}}s_F.$$

  Then we evaluate at the upper limit of p_F, $p_O + l - \hat{p}^C_{F4} = \frac{3l}{2} + \frac{p_{\mathrm{O}}}{2} - 1 - \frac{f}{2} + \frac{s_F}{2}$. To make $p_O + l - \hat{p}^C_{F4} \ge 0$, we get $p_O \le \hat{p}^C_{O13} = f - s_F - 3l + 2$. Then we evaluate at the lower limit of p_F, $\hat{p}^C_{F4} - p_O + l = 1 + \frac{l}{2} - \frac{p_{\mathrm{O}}}{2} + \frac{f}{2} - \frac{s_F}{2}$. To make $\hat{p}^C_{F4} - p_O + l \ge 0$, we get $p_O \le \hat{p}^C_{O14} = f - s_F + l + 2$. Note here, $\hat{p}^C_{O14} - \hat{p}^C_{O13} = 4l$ is positive. When $p_O < \hat{p}^C_{O13}$, solve the Lagrangian

$\pi_{L1FB} = p_F\left(\frac{1}{2}-\frac{l}{4}-\frac{p_F}{4}+\frac{p_{\mathrm{O}}}{4}\right)+\left(f-s_F\right)\left(\frac{l}{4}+\frac{p_F}{4}-\frac{p_{\mathrm{O}}}{4}\right)+\lambda\left(l+p_{\mathrm{O}}-p_F\right)$, we get the boundary solution $p_F^* = \hat{p}_{F3}^C = p_{\mathrm{O}}+l$ and $\pi_{L1FB}^* = \frac{1}{2}l-\frac{1}{2}l^2+\frac{1}{2}p_{\mathrm{O}}-\frac{1}{2}p_{\mathrm{O}}l+\frac{1}{2}lf-\frac{1}{2}ls_F$. When $p_O > \hat{p}_{O14}^C$, solve the Lagrangian $\pi_{L2FB} = p_F\left(\frac{1}{2}-\frac{l}{4}-\frac{p_F}{4}+\frac{p_{\mathrm{O}}}{4}\right)+\left(f-s_F\right)\left(\frac{l}{4}+\frac{p_F}{4}-\frac{p_{\mathrm{O}}}{4}\right)+\lambda\left(p_F-p_{\mathrm{O}}+l\right)$, we get the boundary solution $p_F^* = \hat{p}_{F5}^C = p_{\mathrm{O}}-l$ and $\pi_{L2FB}^* = \frac{p_O-l}{2}$;

- Case C: When $\hat{p}_{O3}^C < p_O \le \hat{p}_{O2}^C$, we get $p_O+l \le p_F < p_O+1$, the total profit function is $\pi_{FC} = p_F\left(\frac{1}{2}-\frac{p_F}{2}+\frac{p_{\mathrm{O}}}{2}\right)+\frac{\left(f-s_F\right)l}{2}$. We derive negative second order derivative $\frac{d^2\pi_{FC}}{dp_F^2}=-1$, so we get $p_F = \hat{p}_{F2}^C = \left(p_O+1\right)/2$ such that $\frac{d\pi_{FC}}{dp_F}=0$. The total profit in this case is $\pi_{FC}^* = \frac{1}{8}+\frac{1}{4}p_{\mathrm{O}}+\frac{1}{8}p_{\mathrm{O}}^2+\frac{1}{2}lf-\frac{1}{2}ls_F$. To reach this optimal price and profit, we need to have $p_O+l \le \hat{p}_{F2}^C < p_O+1$. For the upper limit, $p_O+1-\hat{p}_{F2}^C = \left(p_O+1\right)/2>0$ when $p_O > \hat{p}_{O11}^C = -1$. For the lower limit $\hat{p}_{F2}^C - p_O - l = \frac{1}{2}-\frac{p_{\mathrm{O}}}{2}-l>0$ when $p_O < \hat{p}_{O12}^C = 1-2l$. Notice that $\hat{p}_{O12}^C-\hat{p}_{O11}^C = 2\left(1-l\right)>0$, so we have $\hat{p}_{O11}^C < p_O < \hat{p}_{O12}^C$. Next, we derive the boundary solution when $p_O < \hat{p}_{O11}^C$. We solve the Lagrangian $\pi_{L1FC} = p_F\left(\frac{1}{2}-\frac{p_F}{2}+\frac{p_{\mathrm{O}}}{2}\right)+\frac{\left(f-s_F\right)l}{2}+\lambda\left(1+p_{\mathrm{O}}-p_F\right)$, and get the boundary solution $p_F^* = \hat{p}_{F1}^C = p_{\mathrm{O}}+1$ and $\pi_{L1FC}^* = \frac{\left(f-s_F\right)l}{2}$. Then when $p_O > \hat{p}_{O12}^C$, we solve the Lagrangian $\pi_{L2FC} = p_F\left(\frac{1}{2}-\frac{p_F}{2}+\frac{p_{\mathrm{O}}}{2}\right)+\frac{\left(f-s_F\right)l}{2}+\lambda\left(p_F-p_{\mathrm{O}}-l\right)$, and get the boundary solution $p_F^* = \hat{p}_{F3}^C = p_{\mathrm{O}}+l$ and $\pi_{L2FC}^* = \frac{1}{2}l-\frac{1}{2}l^2+\frac{1}{2}p_{\mathrm{O}}-\frac{1}{2}p_{\mathrm{O}}l+\frac{1}{2}lf-\frac{1}{2}ls_F$;
- Case D: When $p_O \le \hat{p}_{O3}^C$, we get $p_F > p_O+1$, the total profit function is $\pi_{FD} = \left(p_F\right)\cdot a_{FD}^C = 0$. Hence, we have no best response function for this case.

Next, we summarize the offline retailer's overall best response function by consolidating their best response from above.

- Case A: $p_F^* = \hat{p}_{F5}^C = p_{\mathrm{O}}-l$ and the corresponding total profit is π_{FA}^*;

- Case B: When $p_O < \hat{p}^C_{O13}$, the boundary solution is $p^*_F = \hat{p}^C_{F3} = p_O + l$ and the corresponding total profit is π^*_{L1FB}.

  When $\hat{p}^C_{O13} < p_O < \hat{p}^C_{O14}$, the interior solution is $p^*_F = \hat{p}^C_{F4} = \left(p_O + f - s_F - l + 2\right)/2$ and the corresponding total profit is π^*_{FB}.

  When $p_O > \hat{p}^C_{O14}$, the boundary solution is $p^*_F = \hat{p}^C_{F5} = p_O - l$ and the corresponding total profit is π^*_{L2FB};

- Case C: When $p_O < \hat{p}^C_{O11}$, the boundary solution is $p^*_F = \hat{p}^C_{F1} = p_O + 1$ and the corresponding total profit is π^*_{L1FC}.

  When $\hat{p}^C_{O11} < p_O < \hat{p}^C_{O12}$, the interior solution is $p^*_F = \hat{p}^C_{F2} = \left(p_O + 1\right)/2$ and the corresponding total profit is π^*_{FC}.

  When $p_O > \hat{p}^C_{O12}$, the boundary solution is $p^*_F = \hat{p}^C_{F3} = p_O + l$ and the corresponding total profit is π^*_{L2FC}.

From the summary, we find $\pi^*_{FA} = \pi^*_{L2FB}$, so π^*_A is dominated. We also notice that $\pi^*_{L1FB} = \pi^*_{L2FC}$. Hence, we compare the two boundaries $\hat{p}^C_{O13}$ and $\hat{p}^C_{O12}$, and we get $\hat{p}^C_{O13} - \hat{p}^C_{O12} = -l + 1 + f - s_F$. We derive $\hat{p}^C_{O13} > \hat{p}^C_{O12}$ when $f > s_F + l - 1$. Therefore, we have:

- Case F1: $f > \hat{f}_{F1} = s_F + l - 1$

  When $p_O < \hat{p}^C_{O11}$, $p^*_F = \hat{p}^C_{F1}$ and the total profit is π^*_{L1FC}.

  When $\hat{p}^C_{O11} < p_O < \hat{p}^C_{O12}$, $p^*_F = \hat{p}^C_{F2}$ and the total profit is π^*_{FC}.

  When $\hat{p}^C_{O12} < p_O < \hat{p}^C_{O13}$, $p^*_F = \hat{p}^C_{F3}$ and the total profit is π^*_{L1FB}.

  When $\hat{p}^C_{O13} < p_O < \hat{p}^C_{O14}$, $p^*_F = \hat{p}^C_{F4}$ and the total profit is π^*_{FB}.

  When $p_O > \hat{p}^C_{O14}$, $p^*_F = \hat{p}^C_{F5}$ and the total profit is π^*_{L2FB};

When $f < s_F + l - 1$, i.e., $\hat{p}^C_{O13} < \hat{p}^C_{O12}$, we need to compare π^*_{FB} and π^*_{FC}. Hence, we get $\pi^*_{FC} - \pi^*_{FB} = -\frac{1}{8} + \frac{1}{16}p_O^2 + \frac{3}{8}lf - \frac{3}{8}ls_F + \frac{1}{4}l - \frac{1}{16}f^2 + \frac{1}{8}fs_F - \frac{1}{16}s_F^2 + \frac{1}{8}p_O l - \frac{1}{16}l^2 - \frac{1}{4}f + \frac{1}{8}p_O f + \frac{1}{4}s_F - \frac{1}{8}p_O s_F$.

We derive positive second order derivative $\frac{d^2\left(\pi^*_{FC} - \pi^*_{FB}\right)}{dp_O^2} = \frac{1}{8}$. Then we evaluate $\pi^*_{FC} - \pi^*_{FB}$ when $p_O = \hat{p}^C_{O12}$, and we get $\pi^*_{FC} - \pi^*_{FB} = -\frac{\left(-l + 1 + f - s_F\right)^2}{16} < 0$. We evaluate $\pi^*_{FC} - \pi^*_{FB}$ when $p_O = \hat{p}^C_{O13}$, and we get $\pi_{FC}^{*}-\pi_{FB}^{*}=\frac{\left(-l+1+f-s_F\right)^2}{8}>0$. After solving $\pi_{FC}^{*}-\pi_{FB}^{*}=0$, we get two roots $p_{OA}=-\sqrt{2}f+\sqrt{2}l+\sqrt{2}s_F-\sqrt{2}-f-l+s_F$ and $p_{OB}=\sqrt{2}f-\sqrt{2}l-\sqrt{2}s_F+\sqrt{2}-f-l+s_F$. Then to compare p_{OA} and p_{OB}, we take the difference $p_{OA}-p_{OB}=-2\sqrt{2}\left(-l+1+f-s_F\right)$. When $f=s_F+l-1$, we have $p_{OA}-p_{OB}=0$. Since $\frac{d\left(p_{OA}-p_{OB}\right)}{df}=-2\sqrt{2}<0$ and $f<s_F+l-1$, we have $p_{OA}-p_{OB}>0$. Therefore, the smaller root p_{OB} is inside the range and we get $\hat{p}_{O22}^{C}=\sqrt{2}f-\sqrt{2}l-\sqrt{2}s_F+\sqrt{2}-f-l+s_F$. Since $\frac{d\hat{p}_{O22}^{C}}{df}=\sqrt{2}-1>0$, $\hat{p}_{O22}^{C}$ decrease as f decreases. Next, we will compare $\hat{p}_{O22}^{C}$ with $\hat{p}_{O11}^{C}$ and $\hat{p}_{O14}^{C}$. First, we get $\frac{d\hat{p}_{O11}^{C}}{df}=0$ and $\frac{d\hat{p}_{O14}^{C}}{df}=1$. Given $\frac{d\hat{p}_{O14}^{C}}{df}>\frac{d\hat{p}_{O22}^{C}}{df}>\frac{d\hat{p}_{O11}^{C}}{df}$, $\hat{p}_{O22}^{C}$ have a chance to intersect with $\hat{p}_{O11}^{C}$ and $\hat{p}_{O14}^{C}$. Second, let $\hat{p}_{O22}^{C}=\hat{p}_{O11}^{C}$, so we have $f_{11}=3l+s_F-3+2\sqrt{2}l-2\sqrt{2}$. Let $\hat{p}_{O22}^{C}=\hat{p}_{O14}^{C}$, so we have $f_{14}=\hat{f}_{F2}=s_F-(3+2\sqrt{2})l-1$. Then, we compare f_{11} and f_{14} , we get $f_{14}-f_{11}=2\left(3+2\sqrt{2}\right)\left(-l-1+\sqrt{2}\right)$. Note that $f_{14}-f_{11}>0$ when $0<l<\frac{1}{3}$. Hence, when f decreases, $\hat{p}_{O22}^{C}$ will reach $\hat{p}_{O23}^{C}=\hat{p}_{O14}^{C}$ first. Therefore, to summarize, we have:

- Case F2: $\hat{f}_{F2}<f<\hat{f}_{F1}$

  When $p_O\le\hat{p}_{O21}^{C}=\hat{p}_{O11}^{C}$, $p_F^{*}=\hat{p}_{F1}^{C}$ and the total profit is π_{L1FC}^{*}.

  When $\hat{p}_{O21}^{C}<p_O\le\hat{p}_{O22}^{C}$, $p_F^{*}=\hat{p}_{F2}^{C}$ and the total profit is π_{FC}^{*}.

  When $\hat{p}_{O22}^{C}<p_O\le\hat{p}_{O23}^{C}$, $p_F^{*}=\hat{p}_{F4}^{C}$ and the total profit is π_{FB}^{*}.

  When $p_O>\hat{p}_{O23}^{C}$, $p_F^{*}=\hat{p}_{F5}^{C}$ and the total profit is π_{L2FB}^{*};

When $f<\hat{f}_{F2}$, we have $\hat{p}_{O22}^{C}>\hat{p}_{O23}^{C}$, so we need to compare π_{FC}^{*} and π_{L2FB}^{*} . We derive $\pi_{FC}^{*}-\pi_{L2FB}^{*}=\frac{1}{8}-\frac{1}{4}p_O+\frac{1}{8}p_O^2+\frac{1}{2}lf-\frac{1}{2}ls_F+\frac{1}{2}l$ and the second order derivative $\frac{d^2\left(\pi_{FC}^{*}-\pi_{L2FB}^{*}\right)}{dp_O^2}=\frac{1}{4}$ is positive. We first evaluate $\pi_{FC}^{*}-\pi_{L2FB}^{*}$ when $p_O=\hat{p}_{O11}^{C}$, and get $\pi_{FC}^{*}-\pi_{L2FB}^{*}=\frac{1}{2}+\frac{1}{2}lf-\frac{1}{2}ls_F+\frac{1}{2}l$. Then we get $\frac{d\left(\pi_{FC}^{*}-\pi_{L2FB}^{*}\right)}{df}=\frac{l}{2}>0$. When $f=\hat{f}_{F2}$, we have $\pi_{FC}^{*}-\pi_{L2FB}^{*}=\frac{(2\sqrt{2}+3)(-l-1+\sqrt{2})(l-1+\sqrt{2})}{2}>0$, assuming $0<l<\frac{1}{3}$. Let $\pi_{FC}^{*}-\pi_{L2FB}^{*}=0$, we have

$f=\hat{f}_{F3}=s_F-1-\frac{1}{l}$. Hence when $\hat{f}_{F3}<f<\hat{f}_{F2}$, we have $\frac{1}{2}+\frac{1}{2}lf-\frac{1}{2}ls_F+\frac{1}{2}l>0$. Then we evaluate $\frac{d\left(\pi_{FC}^*-\pi_{L2FB}^*\right)}{dp_O}=\frac{p_O}{4}-\frac{1}{4}$ when $p_O=\hat{p}_{O11}^C$, and get $\frac{d\left(\pi_{FC}^*-\pi_{L2FB}^*\right)}{dp_O}=-\frac{1}{2}<0$. Next, we derive the upper boundary of p_O by solving $\pi_{FC}^*-\pi_{L2FB}^*=0$. We get two roots $p_{OA}=1+2\sqrt{-l\left(f-s_F+1\right)}$ and $p_{OB}=1-2\sqrt{-l\left(f-s_F+1\right)}$. Then we compare p_{OA} and p_{OB}, and get $p_{OA}-p_{OB}=4\sqrt{-l\left(f-s_F+1\right)}>0$. So we pick up the smaller root and have $\hat{p}_{O32}^C=p_{OB}=1-2\sqrt{\left(-f+s_F-1\right)l}$. To evaluate $\hat{p}_{O32}^C$, we first have $\frac{d\hat{p}_{O32}^C}{df}=\frac{l}{\sqrt{-lf+ls_F-l}}>0$ and $\frac{d\hat{p}_{O11}^C}{df}=0$. Then we solve $\hat{p}_{O32}^C=\hat{p}_{O11}^C$ and get $f=\hat{f}_{F3}=s_F-1-\frac{1}{l}$. Hence, we have $\hat{p}_{O11}^C<\hat{p}_{O32}^C$. To summarize the case, we have:

- Case F3: $\hat{f}_{F3}<f<\hat{f}_{F2}$

  When $p_O\le\hat{p}_{O31}^C=\hat{p}_{O11}^C$, $p_F^*=\hat{p}_{F1}^C$ and the total profit is π_{L1FC}^*.

  When $\hat{p}_{O31}^C<p_O\le\hat{p}_{O32}^C$, $p_F^*=\hat{p}_{F2}^C$ and the total profit is π_{FC}^*.

  When $p_O>\hat{p}_{O32}^C$, $p_F^*=\hat{p}_{F5}^C$ and the total profit is π_{L2FB}^*;

When $f<\hat{f}_{F3}$, we have $\hat{p}_{O11}^C>\hat{p}_{O32}^C$, so we need to compare π_{L1FC}^* and π_{L2FB}^*. We derive $\pi_{L1FC}^*-\pi_{L2FB}^*=\frac{\left(f-s_F\right)l}{2}-\frac{p_O}{2}+\frac{l}{2}$ and after solving $\pi_{L1FC}^*-\pi_{L2FB}^*=0$, we have $p_O=\hat{p}_{O41}^C=(f-s_F+1)l$. To summarize, we have:

- Case F4: $f\le\hat{f}_{F3}$

  When $p_O\le\hat{p}_{O41}^C$, $p_F^*=\hat{p}_{F1}^C$ and the total profit is π_{L1FC}^*.

  When $p_O>\hat{p}_{O41}^C$, $p_F^*=\hat{p}_{F5}^C$ and the total profit is π_{L2FB}^*.

Now let's derive e-retailer's best response functions p_O^* to the offline retailer's choice of offline price under each case.

- Case A: When $p_O>\hat{p}_{O1}^C$, we get $p_O>p_F+l$, the total profit function is $\pi_{OA}=p_O\cdot\left(a_{EA}^C+a_{SA}^C\right)-f\cdot a_{EA}^C=0$. Hence, there is no best response function in this case.
- Case B: When $\hat{p}_{O2}^C<p_O\le\hat{p}_{O1}^C$, we get $p_F-l<p_O\le p_F+l$, the total profit function is $\pi_{OB}=p_O\cdot\left(a_{EB}^C+a_{SB}^C\right)-f\cdot a_{EB}^C=p_O\left(\frac{l}{4}+\frac{p_F}{4}-\frac{p_O}{4}\right)-f\left(\frac{l}{4}+\frac{p_F}{4}-\frac{p_O}{4}\right)$ and we derive the second

order derivative $\frac{d^2\pi_{OB}}{dp_O^2}=-\frac{1}{2}<0$. Then we solve $\frac{d\pi_{OB}}{dp_O}=0$ and get $p_O^*=\hat{p}_{O2}^C=\left(p_F+f+l\right)/2$ and $\pi_{OB}^*=\frac{\left(-l-p_F+f\right)^2}{16}$. Note that we have the condition $p_F-l<p_O\le p_F+l$, so we first evaluate the lower boundary $p_O-\left(p_F-l\right)$. When $p_O=\left(p_F+f+l\right)/2$, we get $p_O-\left(p_F-l\right)=\frac{3l}{2}-\frac{p_F}{2}+\frac{f}{2}$. We derive negative derivative $\frac{d\left(\frac{3l}{2}-\frac{p_F}{2}+\frac{f}{2}\right)}{dp_F}=-\frac{1}{2}$ and get $p_F=3l+f$ when $\frac{3l}{2}-\frac{p_F}{2}+\frac{f}{2}=0$. Hence, we need to have $p_F<3l+f$. Then we evaluate the upper boundary p_F+l-p_O. When $p_O=\left(p_F+f+l\right)/2$, we get $p_F+l-p_O=\frac{l}{2}+\frac{p_F}{2}-\frac{f}{2}$. We derive positive derivative $\frac{d\left(\frac{l}{2}+\frac{p_F}{2}-\frac{f}{2}\right)}{dp_F}=\frac{1}{2}$ and get $p_F=\hat{p}_{F11}=f-l$ when $\frac{l}{2}+\frac{p_F}{2}-\frac{f}{2}=0$. Hence, we need to have $p_F>f-l$. Then, we check the compatibility and have $\left(3l+f\right)-\left(f-l\right)=4l>0$. So, we need to satisfy the condition $f-l<p_F<3l+f$ in this case. When $p_F<f-l$, solve the Lagrangian $\pi_{L1OB}=p_{\mathrm{O}}\left(\frac{l}{4}+\frac{p_F}{4}-\frac{p_{\mathrm{O}}}{4}\right)-f\left(\frac{l}{4}+\frac{p_F}{4}-\frac{p_{\mathrm{O}}}{4}\right)+\lambda\left(p_F-p_{\mathrm{O}}+l\right)$, we get the boundary solution $p_O^*=\hat{p}_{O1}^C=p_F+l$ and $\pi_{L1OB}^*=0$. When $p_F>3l+f$, we get the boundary solution $p_O^*=p_F-l$ and $\pi_{L2OB}^*=-\frac{l\left(l-p_F+f\right)}{2}$.

- Case C: When $\hat{p}_{O3}^C<p_O\le\hat{p}_{O2}^C$, we get $p_F-1\le p_O<p_F-l$, the total profit function is $\pi_{OC}=p_{\mathrm{O}}\cdot\left(a_{EC}^C+a_{SC}^C\right)-f\cdot a_{EC}^C=\frac{p_{\mathrm{O}}\left(p_F-p_{\mathrm{O}}\right)}{2}-\frac{lf}{2}$ and we derive the second order derivative $\frac{d^2\pi_{OC}}{dp_O^2}=-1<0$. Then we solve $\frac{d\pi_{OC}}{dp_O}=0$ and get $p_O^*=\hat{p}_{O3}^C=\frac{p_F}{2}$ and $\pi_{OC}^*=\frac{p_F^2}{8}-\frac{lf}{2}$. Note that we have the condition $p_F-1\le p_O<p_F-l$, so we first evaluate the lower boundary $p_O-\left(p_F-1\right)$. When $p_O=\frac{p_F}{2}$, we get $p_O-\left(p_F-1\right)=1-\frac{p_F}{2}$. We derive negative derivative $\frac{d\left(1-\frac{p_F}{2}\right)}{dp_F}=-\frac{1}{2}$

and get $p_F = \hat{p}_{F13} = 2$ when $1 - \frac{p_F}{2} = 0$. Hence, we need to have $p_F < 2$. Then we evaluate the upper boundary $p_F - l - p_O$. When $p_O = \frac{p_F}{2}$, we get $p_F - l - p_O = \frac{p_F}{2} - l$. We derive positive derivative $\frac{d\left(\frac{p_F}{2} - l\right)}{dp_F} = \frac{1}{2}$ and get $p_F = 2l$ when $\frac{p_F}{2} - l = 0$. Hence, we need to have $p_F > 2l$. Then, we check the compatibility and have $2 - 2l > 0$ based on our assumption that $0 < l < \frac{1}{3}$. So, we need to satisfy the condition $2l < p_F < 2$ in this case. When $p_F < 2l$, solve the Lagrangian $\pi_{L1OC} = \frac{p_O\left(p_F - p_O\right)}{2} - \frac{lf}{2} + \lambda\left(p_F - p_O - l\right)$, we get the boundary solution $p_O^* = p_F - l$ and $\pi_{L1OC}^* = -\frac{l\left(l - p_F + f\right)}{2}$. When $p_F > 2$, solve the Lagrangian $\pi_{L2OC} = \frac{p_O\left(p_F - p_O\right)}{2} - \frac{lf}{2} + \lambda\left(1 + p_O - p_F\right)$, we get the boundary solution $p_O^* = \hat{p}_{O4}^C = p_F - 1$ and $\pi_{L2OC}^* = -\frac{1}{2} + \frac{p_F}{2} - \frac{lf}{2}$.

- Case D: When $p_O \le \hat{p}_{O3}^C$, we get $p_O < p_F - 1$, the total profit function is $\pi_{FD} = p_O \cdot \left(a_{ED}^C + a_{SD}^C\right) - f \cdot a_{ED}^C = \frac{p_O}{2} - \frac{lf}{2}$. We derive positive derivative $\frac{d\pi_{FD}}{dp_O} = \frac{1}{2}$. Hence, we get the boundary solution $p_O^* = p_F - 1$ and $\pi_{L1OD}^* = -\frac{1}{2} + \frac{p_F}{2} - \frac{lf}{2}$.

Next, we summarize the e-retailer's overall best response function by consolidating their best response from above. First, we notice that $\pi_{L2OC}^* = \pi_{L1OD}^*$, so case D is dominated. Therefore, we have the following:

- Case B: When $p_F < f - l$, the boundary solution is $p_O^* = \hat{p}_{O1}^C = p_F + l$ and the corresponding total profit is π_{L1OB}^*.

  When $f - l < p_F < 3l + f$, the interior solution is $p_O^* = \hat{p}_{O2}^C = \left(p_F + f + l\right)/2$ and the corresponding total profit is π_{OB}^*.

  When $p_F > 3l + f$, the boundary solution is $p_O^* = p_F - l$ and the corresponding total profit is π_{L2OB}^*;
- Case C+D: When $p_F < 2l$, the boundary solution is $p_O^* = p_F - l$ and the corresponding total profit

is π^*_{L1OC}.

When $2l < p_F < 2$, the interior solution is $p^*_O = \hat{p}^C_{O3} = p_F / 2$ and the corresponding total profit is π^*_{OC}.

When $p_F > 2$, the boundary solution is $p^*_O = \hat{p}^C_{O4} = p_F - 1$ and the corresponding total profit is π^*_{L2OC}.

First, we notice that $\pi^*_{L2OB} = \pi^*_{L1OC}$. Then we compare the two boundaries $3l + f$ and $2l$, and we have $3l + f - 2l = l + f > 0$ given $f > 0$. So, we get $2l < 3l + f$. Then we need to discuss the position of the other two boundaries $f - l$ and 2. Since $f - l < 3l + f$, there are two possible positions for $f - l$, i.e., $f - l < 2l < 3l + f$ and $2l < f - l < 3l + f$. Therefore, we look at the two cases separately.

1. Case 1: When $f - l < 2l$, i.e. $f < 3l$, we have $3l + f < 6l$. Since $0 < f < \frac{1}{3}$, we get $3l + f < 2$. Then we compare π^*_{OB} with π^*_{OC}, and we get $\pi^*_{OB} - \pi^*_{OC} = \frac{1}{16} f^2 + \frac{3}{8} lf - \frac{1}{8} p_F f + \frac{1}{16} l^2 + \frac{1}{8} l p_F - \frac{1}{16} p_F^2$. We derive the second order derivative $\frac{d^2\left(\pi^*_{OB} - \pi^*_{OC}\right)}{dp_F^2} = -\frac{1}{8} < 0$. Then when $p_F = 2l$, we get $\pi^*_{OB} - \pi^*_{OC} = \frac{(l+f)^2}{16} > 0$. When $p_F = 3l + f$, we get $\pi^*_{OB} - \pi^*_{OC} = -\frac{(l+f)^2}{8} < 0$. Therefore, we derive two roots $p_{FA} = \sqrt{2} f + \sqrt{2} l - f + l$ and $p_{FB} = -\sqrt{2} f - \sqrt{2} l - f + l$ by solving $\pi^*_{OB} - \pi^*_{OC} = 0$ and we keep the larger root. We have $p_{FA} - p_{FB} = 2\sqrt{2}\left(l + f\right) > 0$, so we keep $\hat{p}_{F12} = p_{FA} = \sqrt{2}(f + l) - f + l$.

   To sum up, we have:

- Case B+C+D: $0 < f < 3l$

  When $p_F \le f - l$, the boundary solution is $p^*_O = \hat{p}^C_{O1} = p_F + l$ and the corresponding total profit is π^*_{L1OB}.

  When $f - l < p_F < \hat{p}_{F12}$, the interior solution is $p^*_O = \hat{p}^C_{O2} = \left(p_F + f + l\right)/2$ and the corresponding total profit is π^*_{OB}.

  When $\hat{p}_{F12} < p_F < 2$, the interior solution is $p^*_O = \hat{p}^C_{O3} = p_F / 2$ and the corresponding total profit is π^*_{OC}.

  When $p_F > 2$, the boundary solution is $p^*_O = \hat{p}^C_{O4} = p_F - 1$ and the corresponding total profit is

π^*_{L2OC}.

2. Case 2: When $f-l>2l$ and $2<f-l$, we have $f>3l$ and $f>l+2$. Since $0<f<\frac{1}{3}$, we get $l+2>3l$. Hence, we only need to keep $f>l+2$. Then we compare π^*_{L1OB} with π^*_{L1OC} when $p_F<2l$, and we get $\pi^*_{L1OC}-\pi^*_{L1OB}=-\frac{l\left(l-p_F+f\right)}{2}$. We derive a positive first order derivative $\frac{d\left(\pi^*_{L1OC}-\pi^*_{L1OB}\right)}{dp_F}=\frac{l}{2}>0$. When $p_F=2l$, we have $\pi^*_{L1OC}-\pi^*_{L1OB}=\frac{l\left(l-f\right)}{2}$. Since $\frac{d\left(\frac{l\left(l-f\right)}{2}\right)}{df}=-1<0$ and $f>l+2$, we have $\frac{l\left(l-f\right)}{2}<0$. Hence, we keep π^*_{L1OB} when $p_F<2l$.

   Then we need to compare π^*_{OC} with π^*_{L1OB} when $2l<p_F<2$. So, we have $\pi^*_{OC}-\pi^*_{L1OB}=\frac{p_F^2}{8}-\frac{lf}{2}$ and get positive second order derivative $\frac{d^2\left(\pi^*_{OC}-\pi^*_{L1OB}\right)}{dp_F^2}=\frac{1}{4}>0$. When $p_F=2l$, we have $\pi^*_{OC}-\pi^*_{L1OB}=\frac{l\left(l-f\right)}{2}<0$ given $f>l+2$. When $p_F=2$, we have $\pi^*_{OC}-\pi^*_{L1OB}=\frac{1-lf}{2}$. Since $\frac{d\left(\frac{1-lf}{2}\right)}{df}=-\frac{l}{2}<0$ and $f>l+2$, so we have $\frac{1-lf}{2}<\frac{1-l^2-2l}{2}$. Recall that $0<l<\frac{1}{3}$, so we have $\frac{1-l^2-2l}{2}>0$. Hence, we have the boundary of f as $\hat{f}_{O2}=\frac{1}{l}$.

   a. Consider $l+2<f<\frac{1}{l}$, then we have $\frac{1-lf}{2}>0$ and $\pi^*_{OC}>\pi^*_{L1OB}$ when $p_F=2$. We have $\pi^*_{OC}-\pi^*_{L1OB}=\frac{p_F^2}{8}-\frac{lf}{2}=0$ when $p_F=\hat{p}_{F21}=2\sqrt{lf}$.

      Then we compare π^*_{L2OC} with π^*_{L1OB} and have $\pi^*_{L2OC}-\pi^*_{L1OB}=\frac{-1+p_F-lf}{2}$. We take the derivative and have $\frac{d\left(\frac{-1+p_F-lf}{2}\right)}{dp_F}=\frac{1}{2}>0$. Then when $p_F=2$, we have $\pi^*_{L2OC}-\pi^*_{L1OB}=\frac{1-lf}{2}>0$. Hence, we get $\frac{-1+p_F-lf}{2}>0$ and thus $\pi^*_{L2OC}>\pi^*_{L1OB}$.

      We also need to compare π^*_{L2OC} with π^*_{OB}. So, we have

$\pi^*_{L2OC}-\pi^*_{OB}=-\frac{1}{2}+\frac{1}{2}p_F-\frac{3}{8}lf-\frac{1}{16}f^2+\frac{1}{8}p_Ff-\frac{1}{16}l^2-\frac{1}{8}lp_F-\frac{1}{16}p_F^2$. We first derive the second order derivative $\frac{d^2\left(\pi^*_{L2OC}-\pi^*_{OB}\right)}{dp_F^2}=-\frac{1}{8}<0$. When $p_F=f-l$, we have $\pi^*_{L2OC}-\pi^*_{OB}=-\frac{1}{2}-\frac{1}{2}l+\frac{1}{2}f-\frac{1}{2}lf$. We take the derivative and get $\frac{d\left(-\frac{1}{2}-\frac{1}{2}l+\frac{1}{2}f-\frac{1}{2}lf\right)}{df}=\frac{1}{2}-\frac{l}{2}>0$. When $f=l+2$, we have $-\frac{1}{2}-\frac{1}{2}l+\frac{1}{2}f-\frac{1}{2}lf=\frac{1}{2}-\frac{l^2}{2}-l>0$ given $0<l<\frac{1}{3}$. Hence, we get $-\frac{1}{2}-\frac{1}{2}l+\frac{1}{2}f-\frac{1}{2}lf>0$. When $p_F=3l+f$, we have $\pi^*_{L2OC}-\pi^*_{OB}=-\frac{(-1+l)(2l+f-1)}{2}$. We take the derivative and get $\frac{d\left(-\frac{(-1+l)(2l+f-1)}{2}\right)}{df}=\frac{1}{2}-\frac{l}{2}>0$. When $f=l+2$, we have $-\frac{(-1+l)(2l+f-1)}{2}=-\frac{(-1+l)(3l+1)}{2}>0$. Hence, we have $-\frac{(-1+l)(2l+f-1)}{2}>0$. In summary, we have $\pi^*_{L2OC}-\pi^*_{OB}>0$.

Then we compare π^*_{L2OC} with π^*_{L2OB} and have $\pi^*_{L2OC}-\pi^*_{L2OB}=\frac{(-1+l)\left(l-p_F+1\right)}{2}$. We derive the derivative and get $\frac{d\left(\pi^*_{L2OC}-\pi^*_{L2OB}\right)}{dp_F}=\frac{1-l}{2}>0$. When $p_F=3l+f$, we have $\pi^*_{L2OC}-\pi^*_{L2OB}=-\frac{(-1+l)(2l+f-1)}{2}$. Notice that we have proven $-\frac{(-1+l)(2l+f-1)}{2}>0$ from above. So, we get $\pi^*_{L2OC}-\pi^*_{L2OB}>0$. To sum up, we have:

- Case B+C+D: $l+2<f<\frac{1}{l}$

When $p_F\le\hat{p}_{F21}$, the boundary solution is $p_O^*=\hat{p}_{O1}^C=p_F+l$ and the corresponding total profit is π^*_{L1OB}.

When $\hat{p}_{F21}<p_F<2$, the interior solution is $p_O^*=\hat{p}_{O3}^C=p_F/2$ and the corresponding total profit is π_{OC}^*.

When $p_F > 2$, the boundary solution is $p_O^* = \hat{p}_{O4}^C = p_F - 1$ and the corresponding total profit is π_{L2OC}^*.

b. Consider $f > \frac{1}{l}$, we have $\pi_{L1OB}^* > \pi_{OC}^*$ when $2l < p_F < 2$. Then we compare π_{L1OB}^* with π_{L2OC}^* and have $\pi_{L1OB}^* - \pi_{L2OC}^* = \frac{1}{2} - \frac{p_F}{2} + \frac{lf}{2}$. When $p_F = 2$, we have $\pi_{L1OB}^* - \pi_{L2OC}^* = -\frac{1}{2} + \frac{lf}{2} > 0$. When $p_F = f - l$, we have $\pi_{L1OB}^* - \pi_{L2OC}^* = \frac{1}{2} + \frac{1}{2}l - \frac{1}{2}f + \frac{1}{2}lf$. We take the derivative and get $\frac{d\left(\frac{1}{2} + \frac{1}{2}l - \frac{1}{2}f + \frac{1}{2}lf\right)}{df} = -\frac{1}{2} + \frac{l}{2} < 0$. When $f = \frac{1}{l}$, we have $\frac{1}{2} + \frac{1}{2}l - \frac{1}{2}f + \frac{1}{2}lf = \frac{l^2 + 2l - 1}{2l} < 0$ given $0 < l < \frac{1}{3}$. Hence, we have $\frac{1}{2} + \frac{1}{2}l - \frac{1}{2}f + \frac{1}{2}lf < 0$. Therefore, we need to find the cutoff by solving $\pi_{L1OB}^* - \pi_{L2OC}^* = 0$ and we have $p_F = \hat{p}_{F31} = fl + 1$. To sum up, we have:

- Case B+C+D: $f > \frac{1}{l}$

  When $p_F < \hat{p}_{F31}$, the boundary solution is $p_O^* = \hat{p}_{O1}^C = p_F + l$ and the corresponding total profit is π_{L1OB}^*.

  When $p_F > \hat{p}_{F31}$, the boundary solution is $p_O^* = \hat{p}_{O4}^C = p_F - 1$ and the corresponding total profit is π_{L2OC}^*.

3. Case 3: When $f - l > 2l$ and $2 > f - l$, we have $3l < f < l + 2$. Then we compare π_{L1OB}^* with π_{L1OC}^*, and we get $\pi_{L1OB}^* - \pi_{L1OC}^* = \frac{l(l - p_F + f)}{2}$. We derive the derivative and have $\frac{d\left(\pi_{L1OB}^* - \pi_{L1OC}^*\right)}{dp_F} = -\frac{l}{2} < 0$. When $p_F = 2l$, we get $\pi_{L1OB}^* - \pi_{L1OC}^* = \frac{l(f - l)}{2}$. When $f = 3l$, we further have $\frac{l(f - l)}{2} = l^2 > 0$. Hence, we have $\pi_{L1OB}^* - \pi_{L1OC}^* > 0$. We also need to compare π_{L1OB}^*

with π_{OC}^*, so we have $\pi_{L1OB}^* - \pi_{OC}^* = -\frac{p_F^2}{8} + \frac{lf}{2}$. We derive the second order derivative $\frac{d^2\left(\pi_{L1OB}^* - \pi_{OC}^*\right)}{dp_F^2} = -\frac{1}{4} < 0$. When $p_F = 2l$, we get $\pi_{L1OB}^* - \pi_{OC}^* = \frac{l\left(f-l\right)}{2}$. We know from above that $\frac{l\left(f-l\right)}{2} > 0$. When $p_F = f - l$, we get $\pi_{L1OB}^* - \pi_{OC}^* = -\frac{1}{8}f^2 + \frac{3}{4}lf - \frac{1}{8}l^2$. We get negative second order derivative $\frac{d^2\left(-\frac{1}{8}f^2 + \frac{3}{4}lf - \frac{1}{8}l^2\right)}{df^2} = -\frac{1}{4} < 0$. When $f = 3l$, we get $-\frac{1}{8}f^2 + \frac{3}{4}lf - \frac{1}{8}l^2 = l^2 > 0$ and when $f = l + 2$, we get $-\frac{1}{8}f^2 + \frac{3}{4}lf - \frac{1}{8}l^2 = l + \frac{1}{2}l^2 - \frac{1}{2} < 0$ given $0 < l < \frac{1}{3}$. Therefore, we need to find the cutoff by solving $\pi_{L1OB}^* - \pi_{OC}^* = 0$ and we have $f_A = (3 + 2\sqrt{2})l$ and $f_B = (3 - 2\sqrt{2})l$. We keep the larger root and we have the boundary of f as $\hat{f}_{O1} = (3 + 2\sqrt{2})l$.

a. Consider $3l < f < (3 + 2\sqrt{2})l$, then we have $-\frac{1}{8}f^2 + \frac{3}{4}lf - \frac{1}{8}l^2 > 0$ and $\pi_{L1OB}^* > \pi_{OC}^*$. Then we compare π_{OB}^* with π_{OC}^* and have $\pi_{OB}^* - \pi_{OC}^* = \frac{1}{16}f^2 + \frac{3}{8}lf - \frac{1}{8}p_F f + \frac{1}{16}l^2 + \frac{1}{8}lp_F - \frac{1}{16}p_F^2$. We derive the second order derivative $\frac{d^2\left(\pi_{OB}^* - \pi_{OC}^*\right)}{dp_F^2} = -\frac{1}{8} < 0$. When $p_F = f - l$, we get $\pi_{OB}^* - \pi_{OC}^* = -\frac{1}{8}f^2 + \frac{3}{4}lf - \frac{1}{8}l^2$. Notice that we have proven $-\frac{1}{8}f^2 + \frac{3}{4}lf - \frac{1}{8}l^2 > 0$ from above. Hence, we have $\pi_{OB}^* - \pi_{OC}^* > 0$ when $p_F = f - l$. When $p_F = 2$, we get $\pi_{OB}^* - \pi_{OC}^* = \frac{1}{16}f^2 + \frac{3}{8}lf - \frac{1}{4}f + \frac{1}{16}l^2 + \frac{1}{4}l - \frac{1}{4}$. We derive the positive second order derivative $\frac{d^2\left(\frac{1}{16}f^2 + \frac{3}{8}lf - \frac{1}{4}f + \frac{1}{16}l^2 + \frac{1}{4}l - \frac{1}{4}\right)}{df^2} = \frac{1}{8} > 0$. When $f = 3l$, we further get $\frac{1}{16}f^2 + \frac{3}{8}lf - \frac{1}{4}f + \frac{1}{16}l^2 + \frac{1}{4}l - \frac{1}{4} = \frac{7}{4}l^2 - \frac{1}{2}l - \frac{1}{4} < 0$ given $0 < l < \frac{1}{3}$. Similarly, when $f = (3 + 2\sqrt{2})l$, it can be simplified as $-\frac{(2\sqrt{2}+3)(-l-1+\sqrt{2})(3l-1+\sqrt{2})}{4} < 0$. Hence, we have $\pi_{OB}^* - \pi_{OC}^* < 0$ when $p_F = 2$. We need to find the cutoff by solving $\pi_{OB}^* - \pi_{OC}^* = 0$ and we have $p_{FA} = \sqrt{2}f + \sqrt{2}l - f + l$ and $p_{FB} = -\sqrt{2}f - \sqrt{2}l - f + l$. We keep the larger root and we have the boundary of p_F as $\hat{p}_{F12} = \sqrt{2}(f + l) - f + l$.

- Case B+C+D: $3l < f < (3 + 2\sqrt{2})l$

  When $p_F \le \hat{p}_{F11}$, the boundary solution is $p_O^* = \hat{p}_{O1}^C = p_F + l$ and the corresponding total profit is π_{L1OB}^*.

  When $\hat{p}_{F11} < p_F < \hat{p}_{F12}$, the interior solution is $p_O^* = \hat{p}_{O2}^C = \left(p_F + f + l\right)/2$ and the corresponding total profit is π_{OB}^*.

  When $\hat{p}_{F12} < p_F < \hat{p}_{F13}$, the interior solution is $p_O^* = \hat{p}_{O3}^C = p_F / 2$ and the corresponding total profit is π_{OC}^*.

  When $p_F > \hat{p}_{F13}$, the boundary solution is $p_O^* = \hat{p}_{O4}^C = p_F - 1$ and the corresponding total profit is π_{L2OC}^*.

b. Consider $(3 + 2\sqrt{2})l < f < l + 2$, we know that $\pi_{L1OB}^* - \pi_{OC}^* < 0$ when $p_F = f - l$. Solving $\pi_{L1OB}^* - \pi_{OC}^* = 0$ and we pick the larger root we get the cutoff of p_{FA} as $\hat{p}_{F21} = 2\sqrt{lf}$. We have calculated a similar case in case 2. So, we can get the following:

- Case B+C+D: $(3 + 2\sqrt{2})l < f < l + 2$

  When $p_F \le \hat{p}_{F21}$, the boundary solution is $p_O^* = \hat{p}_{O1}^C = p_F + l$ and the corresponding total profit is π_{L1OB}^*.

  When $\hat{p}_{F21} < p_F < 2$, the interior solution is $p_O^* = \hat{p}_{O3}^C = p_F / 2$ and the corresponding total profit is π_{OC}^*.

  When $p_F > 2$, the boundary solution is $p_O^* = \hat{p}_{O4}^C = p_F - 1$ and the corresponding total profit is π_{L2OC}^*.

Based on all 3 cases, we summarize the results as shown in main paper.

C. Equilibriums

Based on the results of Lemma 2, we reorganize the best response function of offline retailer and get the following:

- Case F1 (E-S): The boundary solution is $p_F^* = \hat{p}_{F1}^C = p_O + 1$ and the total profit is π_{L1FC}^*
  1. When $f \le \hat{f}_{F3}$ and $p_O \le \hat{p}_{O41}^C$;
  2. When $f > \hat{f}_{F3}$ and $p_O \le \hat{p}_{O31}^C$.
- Case F2 (E-S-F): The interior solution is $p_F^* = \hat{p}_{F2}^C = \left(p_O + 1\right)/2$ and the total profit is π_{FC}^*
  1. When $\hat{f}_{F3} < f \le \hat{f}_{F2}$ and $\hat{p}_{O31}^C < p_O \le \hat{p}_{O32}^C$;
  2. When $\hat{f}_{F2} < f \le \hat{f}_{F1}$ and $\hat{p}_{O21}^C < p_O \le \hat{p}_{O22}^C$;
  3. When $f > \hat{f}_{F1}$ and $\hat{p}_{O11}^C < p_O \le \hat{p}_{O12}^C$.
- Case F3 (E-F: Deter): The boundary solution is $p_F^* = \hat{p}_{F3}^C = p_O + l$ and the total profit is π_{L1FB}^*
  1. When $f > \hat{f}_{F1}$ and $\hat{p}_{O12}^C < p_O \le \hat{p}_{O13}^C$.
- Case F4 (E-F): The interior solution is $p_F^* = \hat{p}_{F4}^C = \left(p_O + f - s_F - l + 2\right)/2$ and the total profit is π_{FB}^*
  1. When $\hat{f}_{F2} < f \le \hat{f}_{F1}$ and $\hat{p}_{O22}^C < p_O \le \hat{p}_{O23}^C$;
  2. When $f > \hat{f}_{F1}$ and $\hat{p}_{O13}^C < p_O \le \hat{p}_{O14}^C$.
- Case F5 (F): The boundary solution is $p_F^* = \hat{p}_{F5}^C = p_O - l$ and the total profit is π_{L2FB}^*
  1. When $f \le \hat{f}_{F3}$ and $p_O > \hat{p}_{O41}^C$;
  2. When $\hat{f}_{F3} < f \le \hat{f}_{F2}$ and $p_O > \hat{p}_{O32}^C$;
  3. When $f > \hat{f}_{F2}$ and $p_O > \hat{p}_{O23}^C$.

Similarly, we reorganize the best response function of e-retailer and get the following:

- Case O1 (F): The boundary solution is $p_O^* = \hat{p}_{O1}^C = p_F + l$ and the total profit is π_{L1OB}^*
  1. When $f < \hat{f}_{O1}$ and $p_F < \hat{p}_{F11}$;
  2. When $\hat{f}_{O1} \le f < \hat{f}_{O2}$ and $p_F < \hat{p}_{F21}$;
  3. When $f \ge \hat{f}_{O2}$ and $p_F < \hat{p}_{F31}$.
- Case O2 (E-F): The interior solution is $p_O^* = \hat{p}_{O2}^C = \left(p_F + f + l\right)/2$ and the total profit is π_{OB}^*
  1. When $f < \hat{f}_{O1}$ and $\hat{p}_{F11} \le p_F < \hat{p}_{F12}$.

- Case O3 (E-S-F): The interior solution is $p_O^* = \hat{p}_{O3}^C = p_F / 2$ and the total profit is π_{OC}^*
  1. When $f < \hat{f}_{O1}$ and $\hat{p}_{F12} \le p_F < \hat{p}_{F13}$;
  2. When $\hat{f}_{O1} \le f < \hat{f}_{O2}$ and $\hat{p}_{F21} \le p_F < \hat{p}_{F22}$.
- Case O4 (E-S): The boundary solution is $p_O^* = \hat{p}_{O4}^C = p_F - 1$ and the total profit is π_{L2OC}^*
  1. When $f < \hat{f}_{O2}$ and $p_F \ge \hat{p}_{F22}$;
  2. When $f \ge \hat{f}_{O2}$ and $p_F \ge \hat{p}_{F31}$.

Then we derive the possible equilibriums:

- **Case F1 (E-S) + O1 (F):**

  There is no feasible for $p_F = p_O + 1$ and $p_O = p_F + l$. Hence, no equilibrium under this case.

- **Case F1 (E-S) + O2 (E-F):**

  Solve $p_F = p_O + 1$ and $p_O = \left(p_F + f + l\right)/2$, which gives $\tilde{p}_F^C = 2 + l + f$ and $\tilde{p}_O^C = 1 + l + f$.

  <u>Case F1-1</u>: $f \le \hat{f}_{F3}$ and $p_O \le \hat{p}_{O41}^C$.

  Compare $\tilde{p}_O^C$ and $\hat{p}_{O41}^C$,

  $$\hat{p}_{O41}^C - \tilde{p}_O^C = fl - ls_F - f - 1.$$

  Take the derivative we have,

  $$\frac{d\left(\hat{p}_{O41}^C - \tilde{p}_O^C\right)}{df} = l - 1 < 0.$$

  When $f = 0$,

  $$\hat{p}_{O41}^C - \tilde{p}_O^C = -ls_F - 1 < 0.$$

  Hence F1-1 is not feasible.

  <u>Case F1-2</u>: $f > \hat{f}_{F3}$ and $p_O \le \hat{p}_{O31}^C$.

  Compare $\tilde{p}_O^C$ and $\hat{p}_{O31}^C$,

  $$\hat{p}_{O31}^C - \tilde{p}_O^C = -2 - l - f < 0.$$

  Hence F1-2 is not feasible.

  There is no equilibrium for this case.

- **Case F1 (E-S) + O3 (E-S-F):**

  Solve $p_F = p_O + 1$ and $p_O = p_F / 2$, which gives $\tilde{p}_F^C = 2$ and $\tilde{p}_O^C = 1$.

  <u>Case F1-1</u>: $f \le \hat{f}_{F3}$ and $p_O \le \hat{p}_{O41}^C$.

  Compare $\tilde{p}_O^C$ and $\hat{p}_{O41}^C$,

$$\hat{p}_{O41}^{C} - \tilde{p}_{O}^{C} = fl - ls_F + l - 1.$$

Take the derivative,

$$\frac{d\left(\hat{p}_{O41}^{C} - \tilde{p}_{O}^{C}\right)}{df} = l > 0.$$

When $f = \hat{f}_{F3}$, $\hat{p}_{O41}^{C} - \tilde{p}_{O}^{C} = -2 < 0$. Hence F1-1 is not feasible.

<u>Case F1-2</u>: $f > \hat{f}_{F3}$ and $p_O \le \hat{p}_{O31}^{C}$.

Compare $\tilde{p}_{O}^{C}$ and $\hat{p}_{O31}^{C}$,

$$\hat{p}_{O31}^{C} - \tilde{p}_{O}^{C} = -2 < 0.$$

Hence F1-2 is not feasible.

There is no equilibrium for this case.

- **Case F1 (E-S) + O4 (E-S):**

Solve $p_F = p_O + 1$ and $p_O = p_F - 1$, which gives $\tilde{p}_{F}^{C} = \tilde{p}_{O}^{C} + 1$ and $\tilde{p}_{O}^{C} = p_O$.

<u>Case F1-1</u>: $f \le \hat{f}_{F3}$ and $p_O \le \hat{p}_{O41}^{C}$.

Compare $\tilde{p}_{O}^{C}$ and $\hat{p}_{O41}^{C}$,

$$\hat{p}_{O41}^{C} - \tilde{p}_{O}^{C} = fl - ls_F + l - p_O.$$

Take the derivative,

$$\frac{d\left(\hat{p}_{O41}^{C} - \tilde{p}_{O}^{C}\right)}{df} = l > 0.$$

When $f = \hat{f}_{F3}$, we get $\hat{p}_{O41}^{C} - \tilde{p}_{O}^{C} = -1 - p_O < 0$. Hence F1-1 is not feasible.

<u>Case F1-2</u>: $f > \hat{f}_{F3}$ and $p_O \le \hat{p}_{O31}^{C}$.

Compare $\tilde{p}_{O}^{C}$ and $\hat{p}_{O31}^{C}$,

$$\hat{p}_{O31}^{C} - \tilde{p}_{O}^{C} = -1 - p_O < 0.$$

Hence F1-2 is not feasible.

There is no equilibrium for this case.

- **Case F2 (E-S-F) + O1 (F):**

Solve $p_F = \left(p_O + 1\right)/2$ and $p_O = p_F + l$, then we get $\tilde{p}_{F}^{C} = l + 1$ and $\tilde{p}_{O}^{C} = 2l + 1$.

<u>Case F2-1</u>: $\hat{f}_{F3} < f \le \hat{f}_{F2}$ and $\hat{p}_{O31}^{C} < p_O \le \hat{p}_{O32}^{C}$.

Compare $\tilde{p}_{O}^{C}$ and $\hat{p}_{O32}^{C}$,

$$\hat{p}_{O32}^{C} - \tilde{p}_{O}^{C} = -2\sqrt{-fl + ls_F - l} - 2l.$$

Take the derivative,

$$\frac{d\left(\hat{p}_{O32}^{C}-\tilde{p}_{O}^{C}\right)}{df}=\frac{l}{\sqrt{-fl+ls_{F}-l}}>0.$$

When $f=\hat{f}_{F2}$, we have

$$\hat{p}_{O32}^{C}-\tilde{p}_{O}^{C}=-2\sqrt{l^{2}(2\sqrt{2}+3)}-2<0.$$

Hence F2-1 is not feasible.

<u>Case F2-2</u>: $\hat{f}_{F2}<f\leq\hat{f}_{F1}$ and $\hat{p}_{O21}^{C}<p_{O}\leq\hat{p}_{O22}^{C}$.

Compare $\tilde{p}_{O}^{C}$ and $\hat{p}_{O22}^{C}$,

$$\hat{p}_{O22}^{C}-\tilde{p}_{O}^{C}=\sqrt{2}\left(f-s_{F}-l+1\right)-f-3l+s_{F}-1.$$

Take the derivative,

$$\frac{d\left(\hat{p}_{O22}^{C}-\tilde{p}_{O}^{C}\right)}{df}=\sqrt{2}-1>0.$$

When $f=\hat{f}_{F1}$, we have

$$\hat{p}_{O22}^{C}-\tilde{p}_{O}^{C}=-4l<0.$$

Hence F2-2 is not feasible.

<u>Case F2-3</u>: $f>\hat{f}_{F1}$ and $\hat{p}_{O11}^{C}<p_{O}\leq\hat{p}_{O12}^{C}$.

Compare $\tilde{p}_{O}^{C}$ and $\hat{p}_{O12}^{C}$,

$$\hat{p}_{O12}^{C}-\tilde{p}_{O}^{C}=-4l<0.$$

Hence F2-3 is not feasible.

There is no equilibrium for this case.

- **Case F2 (E-S-F) + O2 (E-F):**

  Solve $p_{F}=\left(p_{O}+1\right)/2$ and $p_{O}=\left(p_{F}+f+l\right)/2$, which gives $\tilde{p}_{F}^{C}=\left(2+l+f\right)/3$ and $\tilde{p}_{O}^{C}=\left(1+2l+2f\right)/3$.

  <u>Case O2-1</u>: $f<\hat{f}_{O1}$ and $\hat{p}_{F11}\leq p_{F}<\hat{p}_{F12}$.

  Compare $\tilde{p}_{F}^{C}$ and $\hat{p}_{F12}^{C}$,

$$\hat{p}_{F12}^{C}-\tilde{p}_{F}^{C}=\sqrt{2}f+\sqrt{2}l-\frac{4f}{3}+\frac{2l}{3}-\frac{2}{3}.$$

  Take the derivative,

$$\frac{d\left(\hat{p}_{F12}^{C}-\tilde{p}_{F}^{C}\right)}{df}=\sqrt{2}-\frac{4}{3}>0 .$$

To have $\hat{p}_{F12}^{C}-\tilde{p}_{F}^{C}>0$, we have $f>-9\sqrt{2}l+4+3\sqrt{2}-13l$.

Compare $\tilde{p}_{F}^{C}$ and $\hat{p}_{F11}^{C}$,

$$\tilde{p}_{F}^{C}-\hat{p}_{F12}^{C}=\frac{2}{3}+\frac{4l}{3}-\frac{2f}{3} .$$

Take the derivative,

$$\frac{d\left(\tilde{p}_{F}^{C}-\hat{p}_{F12}^{C}\right)}{df}=-\frac{2}{3}<0 .$$

Then we need $f<2l+1$. Note that $f<\hat{f}_{O1}$, so we compare $\hat{f}_{O1}$ with $2l+1$,

$$\hat{f}_{O1}-(2l+1)=(2\sqrt{2}+1)l-1 .$$

Take the derivative,

$$\frac{d\left((2\sqrt{2}+1)l-1\right)}{dl}=2\sqrt{2}+1>0 .$$

When $\hat{f}_{O1}-(2l+1)=0$,

$$l=-\frac{1}{7}+\frac{2\sqrt{2}}{7}\approx 0.26 .$$

Hence when $-\frac{1}{7}+\frac{2\sqrt{2}}{7}<l<\frac{1}{3}$, we have $\hat{f}_{O1}-(2l+1)>0$ and $f<2l+1$.

Next, we check the compatibility with $f>-9\sqrt{2}l+4+3\sqrt{2}-13l$.

When $-\frac{1}{7}+\frac{2\sqrt{2}}{7}<l<\frac{1}{3}$, we have

$$(2l+1)-\left(-9\sqrt{2}l+4+3\sqrt{2}-13l\right)=-3+15l+9\sqrt{2}l-3\sqrt{2}>0 .$$

Hence, case O2-1 is feasible.

When $0<l<-\frac{1}{7}+\frac{2\sqrt{2}}{7}$, we have

$$\hat{f}_{O1}-(2l+1)<0 \text{ and we keep } f<\hat{f}_{O1} .$$

We check the compatibility and have

$$\hat{f}_{O1}-\left(-9\sqrt{2}l+4+3\sqrt{2}-13l\right)=(2\sqrt{2}+3)l+9\sqrt{2}l-4-3\sqrt{2}+13l<0 .$$

Hence $0 < l < -\frac{1}{7} + \frac{2\sqrt{2}}{7}$ is not compatible.

To summarize, case O2-1 is feasible when

$$\max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) < f < 2l + 1 \text{ and } -\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{1}{3}.$$

<u>Case F2-3</u>: $f > \hat{f}_{F1}$ and $\hat{p}_{O11}^{C} < p_O \le \hat{p}_{O12}^{C}$.

Compare $\tilde{p}_O^C$ with the boundary, we have

$$\tilde{p}_O^C - \hat{p}_{O11}^C = \frac{4}{3} + \frac{2l}{3} + \frac{2f}{3} > 0 \text{ and } \hat{p}_{O12}^C - \tilde{p}_O^C = \frac{2}{3} - \frac{8l}{3} - \frac{2f}{3}.$$

Take the derivative,

$$\frac{d\left(\hat{p}_{O12}^C - \tilde{p}_O^C\right)}{df} = -\frac{2}{3} < 0.$$

So we need $f < 1 - 4l$. Note that we also have $\max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) < f < 2l + 1$ and $f > \hat{f}_{F1}$. Therefore we need $\hat{f}_{F1} < 1 - 4l$ and $\max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) < 1 - 4l$.

Since $-\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{1}{3}$.

$$\max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) > 1 - 4l.$$

Hence F2-3 is not feasible.

<u>Case F2-2</u>: $\hat{f}_{F2} < f \le \hat{f}_{F1}$ and $\hat{p}_{O21}^C < p_O \le \hat{p}_{O22}^C$.

Note Case O2-1 is feasible when $\max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) < f < 2l + 1$ and $-\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{1}{3}$. Compare $\tilde{p}_O^C$ with $\hat{p}_{O21}^C$,

$$\tilde{p}_O^C - \hat{p}_{O21}^C = \frac{4}{3} + \frac{2l}{3} + \frac{2f}{3} > 0.$$

Then we evaluate $\tilde{p}_O^C$ with $\hat{p}_{O22}^C$,

$$\hat{p}_{O22}^C - \tilde{p}_O^C = \sqrt{2}f - \sqrt{2}l - \sqrt{2}s_F + \sqrt{2} - \frac{5f}{3} - \frac{5l}{3} + s_F - \frac{1}{3}.$$

Take the derivative,

$$\frac{d\left(\hat{p}_{O22}^C - \tilde{p}_O^C\right)}{df} = \sqrt{2} - \frac{5}{3} < 0.$$

So we need $f < \hat{f} = -\frac{30\sqrt{2}l}{7} - \frac{6\sqrt{2}s_F}{7} + \frac{12\sqrt{2}}{7} - \frac{3s_F}{7} + \frac{13}{7} - \frac{43l}{7}$. Considering the criteria of f in this case, we need to have $\hat{f} > \hat{f}_{F2}$, $\hat{f} > \max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right)$, $\hat{f}_{F1} > \max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right)$ and $2l + 1 > \hat{f}_{F2}$ at the same time.

Compare $\hat{f}$ with $\hat{f}_{F2}$,

$$\hat{f} - \hat{f}_{F2} = -\frac{16\sqrt{2}l}{7} - \frac{6\sqrt{2}s_F}{7} + \frac{12\sqrt{2}}{7} - \frac{10s_F}{7} + \frac{20}{7} - \frac{22l}{7}.$$

Take the derivative,

$$\frac{d\left(\hat{f} - \hat{f}_{F2}\right)}{ds_F} = -\frac{6\sqrt{2}}{7} - \frac{10}{7} < 0.$$

To have $\hat{f} - \hat{f}_{F2} > 0$, we need $s_F < \hat{s}_{F1} = -l - \sqrt{2}l + 2$. Then we check the condition $\hat{f} - \max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) > 0$,

$$\frac{d\left(\hat{f} - \max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right)\right)}{ds_F} = -\frac{6\sqrt{2}}{7} - \frac{3}{7} < 0.$$

For this condition to be valid, we need

$$s_F < \hat{s}_{F2} = -\frac{8\sqrt{2}l}{3} - \frac{11l}{3} + \frac{2\sqrt{2}}{3} + \frac{5}{3} + \frac{\max(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l)}{3} - \frac{2\max(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l)\sqrt{2}}{3}.$$

Then, we get

$$\hat{s}_{F2} - \hat{s}_{F1} = -\frac{(-1 + 2\sqrt{2})\left(3\sqrt{2}l + \max\left(0, \frac{(9\sqrt{2} + 13)(-7l + 3\sqrt{2} - 2)}{7}\right) + 4l - 1\right)}{3} < 0,$$

$$\text{with } -\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{1}{3}.$$

Therefore, we keep $s_F < \hat{s}_{F2}$.

Next, we assume $s_F > \hat{s}_{F2}$,

$$\hat{f}_{F1} - \max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) = l - 1 + s_F - \max\left(0, \frac{(9\sqrt{2} + 13)(-7l + 3\sqrt{2} - 2)}{7}\right).$$

Take the derivative,

$$\frac{d\left(\hat{f}_{F1} - \max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right)\right)}{ds_F} = 1 > 0.$$

To have $\hat{f}_{F1} - \max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) > 0$, we need

$$s_F > \hat{s}_{F3} = -l + 1 + \max\left(0, \frac{(9\sqrt{2}+13)(-7l+3\sqrt{2}-2)}{7}\right).$$

Recall we also have $s_F < \hat{s}_{F2}$, we check

$$\hat{s}_{F2} - \hat{s}_{F3} = -\frac{2(1+\sqrt{2})\left(\max\left(0, \frac{(9\sqrt{2}+13)(-7l+3\sqrt{2}-2)}{7}\right) + 4l - 1\right)}{3} < 0 \text{ with } -\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{1}{3}.$$

Hence, the conditions cannot be met simultaneously, and F2-2 is not feasible.

<u>Case F2-1</u>: $\hat{f}_{F3} < f \le \hat{f}_{F2}$ and $\hat{p}^C_{O31} < p_O \le \hat{p}^C_{O32}$.

Recall case O2-1 is feasible when $\max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) < f < 2l + 1$ and $-\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{1}{3}$. Here we make a mild assumption on the upper bound of s_F to make case F2-1 not feasible.

Consider

$$\max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) - \hat{f}_{F2} = \max(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l) - s_F + 1 + 2\sqrt{2}l + 3l > 0.$$

Take the derivative,

$$\frac{d\left(\max\left(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l\right) - \hat{f}_{F2}\right)}{ds_F} = -1 < 0.$$

Then we need

$$s_F < \hat{s}_F = 2\sqrt{2}l + \max(0, -9\sqrt{2}l + 4 + 3\sqrt{2} - 13l) + 3l + 1$$

Recall $\frac{3\sqrt{2}}{7} - \frac{2}{7} < l < \frac{1}{3}$, we can simply the above as,

$$\hat{s}_{F2} = 2\sqrt{2}l + 1 + 3l .$$

Now we solve two sets of piecewise funtions:

$$lb_{Al} := \begin{cases} 3\sqrt{2}l - \sqrt{2} + 4l - 1 & -\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{3\sqrt{2}}{7} - \frac{2}{7} \\ -\frac{8\sqrt{2}l}{3} - \frac{11l}{3} + \frac{5}{3} + \frac{2\sqrt{2}}{3} & \frac{3\sqrt{2}}{7} - \frac{2}{7} < l < \frac{1}{3} \end{cases}$$

$$ub_{Al} := \begin{cases} -7\sqrt{2}l + 3\sqrt{2} - 10l + 5 & -\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{3\sqrt{2}}{7} - \frac{2}{7} \\ 2\sqrt{2}l + 1 + 3l & \frac{3\sqrt{2}}{7} - \frac{2}{7} < l < \frac{1}{3} \end{cases}$$

And,

$$lb_{Al} := \begin{cases} 3\sqrt{2}l - \sqrt{2} + 4l - 1 & -\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{3\sqrt{2}}{7} - \frac{2}{7} \\ -\frac{8\sqrt{2}l}{3} - \frac{11l}{3} + \frac{5}{3} + \frac{2\sqrt{2}}{3} & \frac{3\sqrt{2}}{7} - \frac{2}{7} < l < \frac{1}{3} \end{cases}$$

$$ub_{Al} := \begin{cases} -7\sqrt{2}l + 3\sqrt{2} - 10l + 5 & -\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{3\sqrt{2}}{7} - \frac{2}{7} \\ 2\sqrt{2}l + 1 + 3l & \frac{3\sqrt{2}}{7} - \frac{2}{7} < l < \frac{1}{3} \end{cases}$$

There is no intersection for either sets of graphs.

Thus, there is no equilibrium for this case.

- **Case F2 (E-S-F) + O3 (E-S-F):**

Solve $p_F = \frac{1}{2} + \frac{p_O}{2}$ and $p_O = \frac{p_F}{2}$, which gives $\tilde{p}_F^C = 2/3$ and $\tilde{p}_O^C = 1/3$.

<u>Case O3-1</u>: $f < \hat{f}_{O1}$ and $\hat{p}_{F12} \le p_F < \hat{p}_{F13}$.

Compare $\hat{p}_{F13}^C$ and $\tilde{p}_F^C$,

$$\hat{p}_{F13}^C - \tilde{p}_F^C = \frac{4}{3} > 0.$$

Compare $\tilde{p}_F^C$ and $\hat{p}_{F12}^C$,

$$\tilde{p}_F^C - \hat{p}_{F12}^C = \frac{2}{3} - \sqrt{2}f - \sqrt{2}l + f - l.$$

Take the derivative,

$$\frac{d\left(\tilde{p}_F^C - \hat{p}_{F12}^C\right)}{df} = -\sqrt{2} + 1 < 0.$$

Solve we have $f = -2\sqrt{2}l - 3l + \frac{2}{3} + \frac{2\sqrt{2}}{3}$.

Hence O3-1 is feasible when $0 < f < \min\left(f_{Ol}, -2\sqrt{2}l - 3l + \frac{2}{3} + \frac{2\sqrt{2}}{3}\right)$.

<u>Case O3-2</u>: $\hat{f}_{O1} < f < \hat{f}_{O2}$ and $\hat{p}_{F21} \le p_F < \hat{p}_{F13}$.

Compare $\hat{p}_{F13}^{C}$ and $\tilde{p}_{F}^{C}$,

$$\hat{p}_{F13}^{C} - \tilde{p}_{F}^{C} = \frac{4}{3} > 0 .$$

Compare $\tilde{p}_{F}^{C}$ and $\hat{p}_{F21}^{C}$,

$$\tilde{p}_{F}^{C} - \hat{p}_{F21}^{C} = \frac{2}{3} - 2\sqrt{l}\sqrt{f} .$$

Take the derivative,

$$\frac{d\left(\tilde{p}_{F}^{C} - \hat{p}_{F21}^{C}\right)}{df} = -\frac{\sqrt{l}}{\sqrt{f}} < 0 .$$

Solve we have $f = \dfrac{1}{9l}$. Compare to $\hat{f}_{O2}$,

$$\hat{f}_{O2} - f = \frac{8}{9l} > 0$$

Hence O3-2 is feasible when $\hat{f}_{O1} < f < \dfrac{1}{9l}$.

Recall O3-1 is feasible when $0 < f < \min\left(f_{Ol}, -2\sqrt{2}l - 3l + \dfrac{2}{3} + \dfrac{2\sqrt{2}}{3}\right)$, we can combine the feasible range for case O3 as,

$$\left\{0 < l < \frac{\sqrt{2}}{3} - \frac{1}{3} \text{ and } 0 < f < \frac{1}{9l}\right\}$$

$$\left\{\frac{\sqrt{2}}{3} - \frac{1}{3} < l < \frac{2\sqrt{2}}{3} - \frac{2}{3} \text{ and } 0 < f < -2\sqrt{2}l - 3l + \frac{2}{3} + \frac{2\sqrt{2}}{3}\right\}.$$

Next, we will discuss <u>Case F2-1</u>: $\hat{f}_{F3} < f < \hat{f}_{F2}$ and $\widehat{p}_{O11} < p_{\text{O}} < \widehat{p}_{\text{O32}}$, <u>Case F2-2</u>: $\hat{f}_{F2} < f < \hat{f}_{F1}$ and $\widehat{p}_{O11} < p_{\text{O}} < \widehat{p}_{\text{O22}}$, and <u>Case F2-3</u>: $f > \hat{f}_{F1}$ and $\widehat{p}_{O11} < p_{\text{O}} < \widehat{p}_{\text{O12}}$, jointly as they share a common boundary.

Compare $\hat{p}_{O}^{C}$ and $\tilde{p}_{O11}^{C}$,

$$\tilde{p}_{O}^{C} - \tilde{p}_{O11}^{C} = \frac{4}{3} > 0 .$$

For <u>Case F2-1</u>, compare $\tilde{p}_{O32}^{C}$ and $\hat{p}_{O}^{C}$, with $f = f_s + s_F$:

$$\frac{d\left(\tilde{p}_{O32}^{C} - \tilde{p}_{O}^{C}\right)}{df} = \frac{l}{\sqrt{-l\left(f_s + 1\right)}} > 0 .$$

Since $\widehat{f}_{F3}-s_F<f_s<\widehat{f}_{F2}-s_F$, we have $\tilde{p}^C_{O32}-\tilde{p}^C_O=-\frac{4}{3}<0$ with $f_s=\widehat{f}_{F3}-s_F$. Solve $\frac{d\left(\tilde{p}^C_{O32}-\tilde{p}^C_O\right)}{df}=0$ we have

$$f_s=-\frac{9l+1}{9l}$$

Hence, F2-1 is feasible when $-\frac{9l+1}{9l}<f_s<\widehat{f}_{F2}-s_F$.

For <u>Case F2-2</u>, compare $\tilde{p}^C_{O22}$ and $\hat{p}^C_O$,

$$\frac{d\left(\tilde{p}^C_{O22}-\tilde{p}^C_O\right)}{df}=-2l+\frac{2}{3}>0\,.$$

Since $\widehat{f}_{F2}-s_F<f_s<\widehat{f}_{F1}-s_F$, we have $\tilde{p}^C_{O22}-\tilde{p}^C_O=-2l+\frac{2}{3}>0$ with $f_s=\widehat{f}_{F1}-s_F$. Solve $\frac{d\left(\tilde{p}^C_{O22}-\tilde{p}^C_O\right)}{df}=0$ we have

$$f_s=2\sqrt{2}l-\frac{2\sqrt{2}}{3}-\frac{5}{3}+3l$$

Hence, F2-2 is feasible when $\max\left(\widehat{f}_{F2}-s_F,2\sqrt{2}l-\frac{2\sqrt{2}}{3}-\frac{5}{3}+3l\right)<f_s<\widehat{f}_{F1}-s_F$.

For <u>Case F2-3</u>, compare $\tilde{p}^C_{O22}$ and $\hat{p}^C_O$,

$$\frac{d\left(\tilde{p}^C_{O22}-\tilde{p}^C_O\right)}{df}=-2l+\frac{2}{3}>0\,.$$

Hence, F2-3 is feasible when $f_s>\widehat{f}_{F1}-s_F$.

Now, consolidate the F2 ranges we have

$$\left\{-\frac{9l+1}{9l}<f_s<\widehat{f}_{F2}-s_F\right\}+\left\{\max\left(\widehat{f}_{F2}-s_F,2\sqrt{2}l-\frac{2\sqrt{2}}{3}-\frac{5}{3}+3l\right)<f_s\right\}$$

Simplify,

$$\max\left(-3l-1-2\sqrt{2}l,2\sqrt{2}l-\frac{2\sqrt{2}}{3}-\frac{5}{3}+3l\right)<f_s$$

Hence, the feasible range for F2 is

$$\left\{0<l<\frac{\sqrt{2}}{3}-\frac{1}{3} \text{ and} f_s>-\frac{9l+1}{9l}\right\}$$

$$\left\{\frac{\sqrt{2}}{3}-\frac{1}{3}<l<\frac{1}{3} \text{ and} f_s>2\sqrt{2}l-\frac{2\sqrt{2}}{3}-\frac{5}{3}+3l\right\}$$

Combine with O3 we have two overlapping regions

$$0<l<\frac{\sqrt{2}}{3}-\frac{1}{3}, 0<f<\frac{1}{9l}, -\frac{9l+1}{9l}<f-s_F$$

$$\frac{\sqrt{2}}{3}-\frac{1}{3}<l<\frac{2\sqrt{2}}{3}-\frac{2}{3}, 0<f<-2\sqrt{2}l-3l+\frac{2}{3}+\frac{2\sqrt{2}}{3}, 2\sqrt{2}l-\frac{2\sqrt{2}}{3}-\frac{5}{3}+3l<f-s_F$$

After adjusting the hierachy, the E-S-F equilibrium exist over the following ranges:

$$\left\{0<l<\frac{\sqrt{2}}{3}-\frac{1}{3} \text{ and } 0<s_F<1+\frac{1}{9l}\right\}: 0<f<\frac{1}{9l}$$

$$\left\{0<l<\frac{\sqrt{2}}{3}-\frac{1}{3} \text{ and } 1+\frac{1}{9l}<s_F<1+\frac{2}{9l}\right\}: s_F-1-\frac{1}{9l}<f<\frac{1}{9l}$$

$$\left\{\frac{\sqrt{2}}{3}-\frac{1}{3}<l<\frac{2\sqrt{2}}{3}-\frac{2}{3} \text{ and } 0<s_F<-2\sqrt{2}l+\frac{2\sqrt{2}}{3}+\frac{5}{3}-3l\right\}: 0<f<-2\sqrt{2}l-3l+\frac{2}{3}+\frac{2\sqrt{2}}{3}$$

$$\left\{\frac{\sqrt{2}}{3}-\frac{1}{3}<l<\frac{2\sqrt{2}}{3}-\frac{2}{3} \text{ and (3.31) } <s_F<-4\sqrt{2}l-6l+\frac{7}{3}+\frac{4\sqrt{2}}{3}\right\}:$$

$$s_F+2\sqrt{2}l-\frac{2\sqrt{2}}{3}-\frac{5}{3}+3l<f<-2\sqrt{2}l-3l+\frac{2}{3}+\frac{2\sqrt{2}}{3}$$

- **Case F2 (E-S-F) + O4 (E-S):**

  Solve $\left\{p_F=\frac{1}{2}+\frac{p_O}{2}, p_O=p_F-1\right\}$, which gives $\left\{\tilde{p}_F^C=0, \tilde{p}_O^C=-1\right\}$.

  Hence there is no equilibrium in this case as price is negative.

- **Case F3 (E-F: Deter) + O1 (F):**

  There is no solution to $\left\{p_F=p_O+l, p_O=p_F+l\right\}$, indicating that there is no intersection.

  Hence there is no equilibrium in this case.

- **Case F3 (E-F: Deter) + O2 (E-F):**

  Solve $\left\{p_F=p_O+l, p_O=\frac{l}{2}+\frac{p_F}{2}+\frac{f}{2}\right\}$, which gives $\left\{\tilde{p}_F^C=3l+f, \tilde{p}_O^C=2l+f\right\}$.

  <u>Case O2-1</u>: $0<f<\hat{f}_{O1}$ and $\hat{p}_{F11}\le p_F<\hat{p}_{F12}$.

  Compare $\hat{p}_{F12}^C$ and $\tilde{p}_F^C$,

$$\hat{p}_{F12}^{C} - \tilde{p}_{F}^{C} = (2-\sqrt{2})(-f-l) < 0 .$$

Hence O2-1 has no feasible range and there is no equilibrium in this case.

- **Case F3 (E-F: Deter) + O3 (E-S-F):**

Solve $\left\{ p_F = p_O + l, p_O = \frac{p_F}{2} \right\}$, which gives $\left\{ \tilde{p}_F^C = 2l, \tilde{p}_O^C = l \right\}$.

Case O3-1: $0 < f < \hat{f}_{O1}$ and $\hat{p}_{F12} \le p_F < \hat{p}_{F13}$.

Compare $\hat{p}_{F13}^{C}$ and $\tilde{p}_F^C$,

$$\hat{p}_{F12}^{C} - \tilde{p}_{F}^{C} = (2-\sqrt{2})(-f-l) < 0 .$$

Hence this O3-1 has no feasible range.

Case O3-2: $\hat{f}_{O1} < f < \hat{f}_{O2}$ and $\hat{p}_{F21} \le p_F < \hat{p}_{F13}$.

Compare $\hat{p}_F^C$ and $\tilde{p}_{F21}^{C}$,

$$\hat{p}_{F12}^{C} - \tilde{p}_{F}^{C} = (2-\sqrt{2})(-f-l) < 0 .$$

Hence this O3-1 has no feasible range.

Compare $\tilde{p}_F^C$ and $\hat{p}_{F12}^{C}$,

$$\frac{d\left(\hat{p}_F^C - \tilde{p}_{F12}^{C}\right)}{df} = 2l - 2\sqrt{l}\sqrt{f} .$$

When $f = \hat{f}_{O1}$

$$2l - 2\sqrt{l}\sqrt{f} = -\sqrt{2}l < 0 .$$

Hence this O3-2 has no feasible range.

And there is no equilibrium in this case.

- **Case F3 (E-F: Deter) + O4 (E-S):**

There is no solution to $\left\{ p_F = p_O + l, p_O = p_F - 1 \right\}$, indicating that there is no intersection.

Hence there is no equilibrium in this case.

- **Case F4 (E-F) + O1 (F):**

Solve $\left\{ p_F = 1 - \frac{l}{2} + \frac{p_O}{2} + \frac{f}{2} - \frac{s_F}{2}, p_O = p_F + l \right\}$, which gives

$$\left\{ \tilde{p}_F^C = 2 + f - s_F, \tilde{p}_O^C = 2 + l + f - s_F \right\}.$$

Case O2-1: $0 < f < \hat{f}_{O1}$ and $p_F < \hat{p}_{F11}$.

Compare $\hat{p}_{F11}^{C}$ and $\tilde{p}_{F}^{C}$,

$$\hat{p}_{F13}^{C} - \tilde{p}_{F}^{C} = -l - 2 + s_F .$$

Solve for s_F

$$s_F = l + 2$$

Then O1-1 is feasible when $s_F > l + 2$.

<u>Case O1-2</u>: $\hat{f}_{O1} < f < \hat{f}_{O2}$ and $p_F < \hat{p}_{F21}$.

To compare $\hat{p}_{F21}^{C}$ and $\tilde{p}_{F}^{C}$, first take the second derivative,

$$\frac{d^2\left(\hat{p}_{F21}^{C} - \hat{p}_{F}^{C}\right)}{df^2} = -\frac{\sqrt{l}}{2f^{3/2}} < 0 .$$

Then by checking boundaires,

$$\frac{d\left(\hat{p}_{F21}^{C} - \hat{p}_{F}^{C}\right)}{df} = \frac{\sqrt{l}}{\sqrt{f}} - 1 < 0 .$$

Then we check $\hat{p}_{F21}^{C} - \tilde{p}_{F}^{C}$ at boundaries point $f = \hat{f}_{O1}$,

$$\hat{p}_{F21}^{C} - \tilde{p}_{F}^{C} = -l - 2 + s_F ,$$

and at boundaries point $f = \hat{f}_{O2}$,

$$\hat{p}_{F21}^{C} - \tilde{p}_{F}^{C} = \frac{ls_F - 1}{l} .$$

Now the difference is

$$-l - 2 + s_F - \frac{ls_F - 1}{l} = -l - 2 + s_F$$

Hence in O1-2 is feasible when $s_F > l + 2$.

<u>Case O1-3</u>: $f > \hat{f}_{O2}$ and $p_F < \hat{p}_{F31}$.

To compare $\hat{p}_{F31}^{C}$ and $\tilde{p}_{F}^{C}$, first take the second derivative,

$$\frac{d\left(\hat{p}_{F31}^{C} - \hat{p}_{F}^{C}\right)}{df} = l - 1 < 0 .$$

Then check boundairy, $f = \hat{f}_{O2}$

$$fl - f + s_F - 1 = -\frac{1}{l} + s_F .$$

Then we check $\hat{p}_{F21}^{C} - \tilde{p}_{F}^{C}$ at boundaries point $f = \hat{f}_{O1}$,

$$\hat{p}_{F21}^{C} - \tilde{p}_{F}^{C} = -l - 2 + s_F ,$$

Solve for s_F,

$$s_F = \frac{1}{l} .$$

Hence in O1-3d is feasible when $s_F < l + 2$.

To combine O1 subcases, we notice that when $s_F < l + 2$, none of O1-1, O1-2 and O1-3 will be feasible. Thus, there is no equilibrium in this case.

- **Case F4 (E-F) + O2 (F-F):**

Solve $\left\{ p_F = 1 - \frac{l}{2} + \frac{p_O}{2} + \frac{f}{2} - \frac{s_F}{2}, p_O = \frac{l}{2} + \frac{p_F}{2} + \frac{f}{2} \right\}$,

which gives $\left\{ \tilde{p}_F^C = -\frac{l}{3} + f + \frac{4}{3} - \frac{2s_F}{3}, \tilde{p}_O^C = \frac{l}{3} + f + \frac{2}{3} - \frac{s_F}{3} \right\}$.

<u>Case O2-1</u>: $0 < f < \hat{f}_{O1}$ and $\hat{p}_{F11} < p_F < \hat{p}_{F12}$.

Compare $\hat{p}_F^C$ and $\tilde{p}_{F11}^C$,

$$\hat{p}_F^C - \tilde{p}_{F11}^C = \frac{2l}{3} + \frac{4}{3} - \frac{2s_F}{3} .$$

Solve for s_F,

$$s_F = -\frac{2}{3} < 0 .$$

Note that we assume $s_F < l + 2$, hence we need $\frac{2l}{3} + \frac{4}{3} - \frac{2s_F}{3} > 0$.

Solve for f,

$$f = \frac{5\sqrt{2}l}{3} + \frac{2s_F}{3} + \frac{\sqrt{2}s_F}{3} - \frac{2\sqrt{2}}{3} - \frac{4}{3} + \frac{7l}{3}$$

Compare $\hat{p}_{F12}^C$ and $\tilde{p}_F^C$, take the derivative,

$$\frac{d\left(\hat{p}_{F12}^C - \tilde{p}_F^C\right)}{df} = \sqrt{2} - 2 < 0 .$$

When $f = \widehat{f}_{O1}$,

$$\frac{d\left(\hat{p}_{F12}^C - \tilde{p}_F^C\right)}{ds_F} = \frac{2}{3} > 0 .$$

Hence, we need $\hat{p}_{F12}^C - \tilde{p}_F^C = -\frac{2l}{3} - \frac{4}{3} + \frac{2s_F}{3} < 0$.

Solve for s_F for the previous case, we need

$$s_F > 2 - \frac{3\sqrt{2}l}{2} - 2l .$$

Hence O2-1 is feaible when

$s_F > 2 - \frac{3\sqrt{2}l}{2} - 2l$ and $0 < f < \frac{5\sqrt{2}l}{3} + \frac{2s_F}{3} + \frac{\sqrt{2}s_F}{3} - \frac{2\sqrt{2}}{3} - \frac{4}{3} + \frac{7l}{3}$.

Next, we will discuss <u>Case F4-1</u>: $\hat{f}_{F2} < f < \hat{f}_{F1}$ and $\hat{p}_{O22} < p_O < \hat{p}_{O14}$ and <u>Case F4-2</u>: $f > \hat{f}_{F1}$ and $\hat{p}_{O13} < p_O < \hat{p}_{O14}$ together.

Compare $\hat{p}^C_{O14}$ and $\tilde{p}^C_O$,

$$\hat{p}^C_{O14} - \tilde{p}^C_O = \frac{2l}{3} + \frac{4}{3} - \frac{2s_F}{3} > 0 .$$

Compare $\tilde{p}^C_O$ and $\hat{p}^C_{O22}$,

$$\tilde{p}^C_O - \hat{p}^C_{O22} = \frac{4l}{3} + 2f + \frac{2}{3} - \frac{4s_F}{3} - \sqrt{2}f + \sqrt{2}l + \sqrt{2}s_F - \sqrt{2} .$$

Take the derivative,

$$\frac{d\left(\tilde{p}^C_O - \hat{p}^C_{O22}\right)}{df} = 2 - \sqrt{2} > 0 .$$

Solve then we have $f = \frac{s_F}{3} + \frac{1}{3} - \frac{5\sqrt{2}l}{3} - \frac{\sqrt{2}s_F}{3} + \frac{2\sqrt{2}}{3} - \frac{7l}{3}$.

Since $\frac{s_F}{3} + \frac{1}{3} - \frac{5\sqrt{2}l}{3} - \frac{\sqrt{2}s_F}{3} + \frac{2\sqrt{2}}{3} - \frac{7l}{3} > \hat{f}_{F2}$, solve for s_F we have

$$s_F = -5l + 2$$

So we need $s_F > -5l + 2$ and $\frac{s_F}{3} + \frac{1}{3} - \frac{5\sqrt{2}l}{3} - \frac{\sqrt{2}s_F}{3} + \frac{2\sqrt{2}}{3} - \frac{7l}{3} < f < \hat{f}_{F1}$ to make F4-1 feasible.

And this also satisfies F4-2, when $f > \hat{f}_{F1}$ and $\hat{p}_{O13} < p_O < \hat{p}_{O14}$.

Hence either F4-1 or F4-2 is feaible when $s_F > -5l + 2$ and $\frac{s_F}{3} + \frac{1}{3} - \frac{5\sqrt{2}l}{3} - \frac{\sqrt{2}s_F}{3} + \frac{2\sqrt{2}}{3} - \frac{7l}{3} < f$.

Next, we consolidate the feasible range of O2 and F4 using two piecewise functions,

$$lb_{EF} := \begin{cases} \frac{11}{7} - \frac{18\sqrt{2}l}{7} + \frac{6\sqrt{2}}{7} - \frac{26l}{7} & -\frac{5}{7} + \frac{4\sqrt{2}}{7} < l < 3\sqrt{2} - 4 \\ 2 - \frac{3\sqrt{2}l}{2} - 2l & 3\sqrt{2} - 4 < l < \frac{1}{3} \end{cases}$$

$$ub_{EF} := \begin{cases} l+2 & -\frac{5}{7}+\frac{4\sqrt{2}}{7} < l < \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}.$$

Thus, the E-F equilibrium in this case exist when:

$$\left\{-\frac{5}{7}+\frac{4\sqrt{2}}{7} < l < 3\sqrt{2}-4 \text{ and } \frac{11}{7}-\frac{18\sqrt{2}l}{7}+\frac{6\sqrt{2}}{7}-\frac{26l}{7} < s_F < l+2\right\}$$
$$+\left\{3\sqrt{2}-4 < l < \frac{1}{3} \text{ and } 2-\frac{3\sqrt{2}l}{2}-2l < s_F < l+2\right\}$$
$$:\frac{s_F}{3}+\frac{1}{3}-\frac{5\sqrt{2}l}{3}-\frac{\sqrt{2}s_F}{3}+\frac{2\sqrt{2}}{3}-\frac{7l}{3} < f < \frac{5\sqrt{2}l}{3}+\frac{2s_F}{3}+\frac{\sqrt{2}s_F}{3}-\frac{2\sqrt{2}}{3}-\frac{4}{3}+\frac{7l}{3}$$

- **Case F4 (E-F) + O3 (E-S-F):**

Solve $\left\{p_F = 1-\frac{l}{2}+\frac{p_O}{2}+\frac{f}{2}-\frac{s_F}{2}, p_O = \frac{p_F}{2}\right\}$,

which gives $\left\{\tilde{p}_F^C = \frac{4}{3}-\frac{2l}{3}+\frac{2f}{3}-\frac{2s_F}{3}, \tilde{p}_O^C = \frac{2}{3}-\frac{l}{3}+\frac{f}{3}-\frac{s_F}{3}\right\}$.

<u>Case F4-1</u>: $\hat{f}_{F2} < f < \hat{f}_{F1}$ and $\hat{p}_{O22} \le p_O < \hat{p}_{O14}$.

Compare $\tilde{p}_{O14}^C$ and $\tilde{p}_O^C$,

$$\tilde{p}_{O14}^C - \tilde{p}_O^C = \frac{4}{3}+\frac{4l}{3}+\frac{2f}{3}-\frac{2s_F}{3} > 0.$$

Compare $\tilde{p}_O^C$ and $\tilde{p}_{O22}^C$,

$$\tilde{p}_O^C - \tilde{p}_{O22}^C = \frac{2}{3}+\frac{2l}{3}+\frac{4f}{3}-\frac{4s_F}{3}-\sqrt{2}f+\sqrt{2}l+\sqrt{2}s_F-\sqrt{2} < 0.$$

When $f = \hat{f}_{F1}$,

$$\tilde{p}_O^C - \tilde{p}_{O22}^C = -\frac{2}{3}+2l < 0$$

Solve for f

$$f = s_F - 5 + 9\sqrt{2}l - 3\sqrt{2} + 13l.$$

So we need $\hat{f}_{F2} < f < s_F - 5 + 9\sqrt{2}l - 3\sqrt{2} + 13l$.

Hence, F4-1 is feasible when $-\frac{1}{7}+\frac{2\sqrt{2}}{7} < l < \frac{1}{3}$ and $f_{F2} < f < s_F - 5 + 9\sqrt{2}l - 3\sqrt{2} + 13l$.

<u>Case F4-2</u>: $f > \hat{f}_{F1}$ and $\hat{p}_{O13} \le p_O < \hat{p}_{O14}$.

Compare $\tilde{p}_O^C$ and $\tilde{p}_{O13}^C$, and take the derivative

$$\frac{d\left(\tilde{p}_O^C - \tilde{p}_{O13}^C\right)}{df} = -\frac{2}{3} < 0 .$$

When $f = \widehat{f}_{F1}$,

$$\tilde{p}_O^C - \tilde{p}_{O13}^C = \frac{2}{3}(-1+3l) < 0$$

Hence F4-2 cannot be feasible.

<u>Case O3-1</u>: $0 < f < \hat{f}_{O1}$ and $\hat{p}_{F12} \le p_F < \hat{p}_{F13}$.

Compare $\tilde{p}_{F13}^C$ and $\tilde{p}_F^C$,

$$\tilde{p}_{F13}^C - \tilde{p}_F^C = \frac{2}{3} + \frac{2l}{3} - \frac{2f}{3} + \frac{2s_F}{3} .$$

Solve for f,

$$f = 1 + l + s_F .$$

So we need $-\frac{1}{7} + \frac{2\sqrt{2}}{7} < l < \frac{1}{3}$ and $\widehat{f}_{F2} < f < s_F - 5 + 9\sqrt{2}l - 3\sqrt{2} + 13l$. And it is satisfied.

Compare $\tilde{p}_F^C$ and $\tilde{p}_{F12}^C$,

$$\tilde{p}_F^C - \tilde{p}_{F12}^C = \frac{4}{3} - \frac{5l}{3} + \frac{5f}{3} - \frac{2s_F}{3} - \sqrt{2}f - \sqrt{2}l .$$

Take derivative we have $\frac{5}{3} - \sqrt{2} > 0$.

When $f = s_F - 5 + 9\sqrt{2}l - 3\sqrt{2} + 13l$,

$$\tilde{p}_{F13}^C - \tilde{p}_F^C = -\sqrt{2}s_F + 2l + s_F - 1 + \sqrt{2}l$$

Take derivative on s_F we have $-\sqrt{2} + 1 < 0$.

Solve using two sets of piecewise functions:

$$ub_{A4} := \begin{cases} l+2 & 0 < l < \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$$

$$l-1/l \ + \ s\_\_F \begin{cases} 0 & 0 < l < 1 - \frac{\sqrt{2}}{2} \\ 3\sqrt{2}l - \sqrt{2} + 4l - 1 & 1 - \frac{\sqrt{2}}{2} < l < \frac{1}{3} \end{cases}$$

We have $\frac{4}{3} - \frac{5l}{3} + \frac{5f}{3} - \frac{2s_F}{3} - \sqrt{2}f - \sqrt{2}l$ to be negative, hence O3-1 cannot be feasible.

Case O3-2: $\hat{f}_{O1} < f < \hat{f}_{O2}$ and $\hat{p}_{F21} \le p_F < \hat{p}_{F13}$.

Here we can show that there is no overlapping region between the feasible ranges $\hat{f}_{O1} < f < \hat{f}_{O2}$ and $\widehat{f}_{F2} < f < s_F - 5 + 9\sqrt{2}l - 3\sqrt{2} + 13l$.

Specifically,

$$\hat{f}_{O1} - s_F + 5 - 9\sqrt{2}l + 3\sqrt{2} - 13l = (2\sqrt{2}+3)l - s_F + 5 - 9\sqrt{2}l + 3\sqrt{2} - 13l .$$

Take derivative of s_F, we have $-1 < 0$ and this result satisfies the assumptions.

Hence O3-2 cannot be feasible.

Overall, there is no equilibrium in this case.

- **Case F4 (E-F) + O4 (E-S):**

Solve $\left\{ p_F = 1 - \frac{l}{2} + \frac{p_{\mathrm{O}}}{2} + \frac{f}{2} - \frac{s_F}{2}, p_{\mathrm{O}} = p_F - 1 \right\}$,

which gives $\left\{ \tilde{p}_F^C = -l + f - s_F + 1, \tilde{p}_O^C = -l + f - s_F \right\}$.

Case O4-1: $\hat{f}_{F2} < f < \hat{f}_{F1}$ and $\hat{p}_{O22} \le p_O < \hat{p}_{O14}$.

Compare $\hat{p}_{O14}^C$ and $\tilde{p}_O^C$,

$$\hat{p}_{O14}^C - \tilde{p}_O^C = 2 + 2l > 0 .$$

Take the derivative,

$$\frac{d\left(\dfrac{\hat{p}_{O14}^C - \tilde{p}_O^C}{2-\sqrt{2})}\right)}{df} = 1 > 0 .$$

Check the boundary at $f = \hat{f}_{F1}$,

$$\frac{\hat{p}_{O14}^C - \tilde{p}_O^C}{2-\sqrt{2})} = (2+\sqrt{2})(l-1) < 0 .$$

Hence O4-1 cannot be feasible.

Case O4-2: $f > \hat{f}_{F1}$ and $\hat{p}_{O13} \le p_O < \hat{p}_{O14}$.

Compare $\hat{p}_O^C$ and $\tilde{p}_{O13}^C$,

$$\frac{\hat{p}_O^C - \tilde{p}_{O13}^C}{2-\sqrt{2})} = (2+\sqrt{2})(l-1) < 0 .$$

Take the derivative,

$$\frac{d\left(\frac{\hat{p}^C_{O14}-\tilde{p}^C_O}{2-\sqrt{2})}\right)}{df}=1>0\,.$$

Hence O4-2 cannot be feasible.

There is no equilibrium in this case.

- **Case F5 (F) + O1 (F):**

Solve $\{p_F = p_O - l, p_O = p_F + l\}$, which gives $\{\tilde{p}^C_F = \tilde{p}^C_O - l, \tilde{p}^C_O = \tilde{p}^C_O\}$.

This suggests that two OBFs coincides. We need to derive the range of p_O.

We can directly observe that for <u>Case F5-1</u>: $f < \hat{f}_{F3}$ and $p_O > \hat{p}_{O41}$, <u>Case F5-2</u>: $\hat{f}_{F3} < f < \hat{f}_{F2}$ and $p_O > \hat{p}_{O32}$, and <u>Case F5-3</u>: $f > \hat{f}_{F2}$ and $p_O > \hat{p}_{O23}$.

For O cases, <u>Case O1-1</u>: $0 < f < \hat{f}_{O1}$ and $p_F < \hat{p}_{F11}$.

Compare $\hat{p}^C_{F11}$ and $\tilde{p}^C_F$,

$$\hat{p}^C_{F11} - \tilde{p}^C_F = f - p_O\,.$$

Solve for p_O we have $p_O = f$. So O1-1 is feasible when $0 < f < \hat{f}_{O1}$ and $p_O < f$.

<u>Case O1-2</u>: $\hat{f}_{O1} < f < \hat{f}_{O2}$ and $p_F < \hat{p}_{F21}$.

Compare $\hat{p}^C_{F21}$ and $\tilde{p}^C_F$,

$$\hat{p}^C_{F21} - \tilde{p}^C_F = 2\sqrt{l}\sqrt{f} - p_O + l\,.$$

Solve for p_O we have $p_O = 2\sqrt{l}\sqrt{f} + l$. So O1-2 is feasible when $\hat{f}_{O1} < f < \hat{f}_{O2}$ and $p_O < 2\sqrt{l}\sqrt{f} + l$.

<u>Case O1-3</u>: $f > \hat{f}_{O2}$ and $p_F < \hat{p}_{F31}$.

Compare $\hat{p}^C_{F31}$ and $\tilde{p}^C_F$,

$$\hat{p}^C_{F21} - \tilde{p}^C_F = fl + l - p_O + 1\,.$$

Solve for p_O we have $p_O = fl + l + 1$. So O1-2 is feasible when $f > \hat{f}_{O2}$ and $p_O < fl + l + 1$.

Next, we rule out F5-1. Under F5-1, we have $f < \hat{f}_{F3}$. Take the derivative,

$$\frac{d\hat{f}_{F3}}{ds_F} = 1 > 0\,.$$

Since we have $s_F < l + 2$,

$$\hat{f}_{F3} = s_F - 1 - \frac{1}{l} < 0 .$$

So, we can rule out F5-1.

Now we have Case F5-2: feasible when $f < \hat{f}_{F2}$ and $p_O > \hat{p}_{O32}$, Case F5-3: feasible when $f > \hat{f}_{F2}$ and $p_O > \hat{p}_{O23}$, Case O1-1: $0 < f < \hat{f}_{O1}$ and $p_O < f$, Case O1-2: $\hat{f}_{O1} < f < \hat{f}_{O2}$ and $p_O < 2\sqrt{l}\sqrt{f} + l$, and Case O1-3: $f > \hat{f}_{O2}$ and $p_O < fl + l + 1$.

Now when $s_F = 0$,

$$\hat{f}_{F2} = -3l - 1 - 2\sqrt{2l} .$$

Take derivative,

$$\frac{d\hat{f}_{F2}}{ds_F} = 1 > 0$$

When $s_F = l + 2$,

$$\hat{f}_{F2} = -2l + 1 - 2\sqrt{2l}$$

Then we futher discuss three cases A, B, C.

Case A: $\hat{f}_{F2} < 0$.

For O1-1, we check the compatibility $\hat{p}_{O23} < p_O < f$, under $0 < f < \hat{f}_{O1}$.

$$f - \hat{p}_{O23} = -l - 2 + s_F < 0$$

So O1-1 cannot be feasible.

For O1-2, we check the compatibility $\hat{p}_{O23} < p_O < 2\sqrt{l}\sqrt{f} + l$, under $\hat{f}_{O1} < f < \hat{f}_{O2}$.

So O1-2 cannot be feasible.

Case B: $\hat{f}_{F2} < 0$.

Case C: $\hat{f}_{F2} < 0$.

Case O3-1: $f < \hat{f}_{O1}$ and $\hat{p}_{F12} \le p_F < \hat{p}_{F13}$.

Compare $\hat{p}^C_{F13}$ and $\tilde{p}^C_F$,

$$\hat{p}^C_{F13} - \tilde{p}^C_F = \frac{4}{3} > 0 .$$

Compare $\tilde{p}^C_F$ and $\hat{p}^C_{F12}$,

$$\tilde{p}^C_F - \hat{p}^C_{F12} = \frac{2}{3} - \sqrt{2}f - \sqrt{2}l + f - l .$$

Take the derivative,

$$\frac{d\left(\tilde{p}_F^C - \hat{p}_{F12}^C\right)}{df} = -\sqrt{2} + 1 < 0 .$$

Solve we have $f = -2\sqrt{2}l - 3l + \frac{2}{3} + \frac{2\sqrt{2}}{3}$.

Hence O3-1 is feasible when $0 < f < \min\left(f_{Ol}, -2\sqrt{2}l - 3l + \frac{2}{3} + \frac{2\sqrt{2}}{3} \right)$.

There is no equilibrium in this case.

- **Case F5 (F) + O2 (E-F):**

Solve $\left\{ p_F = p_{\mathrm{O}} - l, p_{\mathrm{O}} = \frac{l}{2} + \frac{p_F}{2} + \frac{f}{2} \right\}$, which gives $\left\{ \tilde{p}_F^C = -l + f, \tilde{p}_O^C = f \right\}$.

Case F5-1: $f < \hat{f}_{F3}$ and $p_O > \hat{p}_{O41}$.

Compare $\tilde{p}_O^C$ and $\tilde{p}_{O41}^C$, then take derivative,

$$\frac{d\left(\tilde{p}_O^C - \tilde{p}_{O41}^C\right)}{df} = -l + 1 > 0 .$$

When $f = \widehat{f}_{F3}$

$$\tilde{p}_O^C - \tilde{p}_{O41}^C = -\frac{1}{l} + s_F$$

Take derivative for s_F we have $1 > 0$.

So we have $-\frac{1}{l} + s_F$ to be negative.

Hence F5-1 cannot be feasible.

Case F5-2: $\hat{f}_{F3} < f < \hat{f}_{F2}$ and $p_O > \hat{p}_{O32}$.

Compare $\tilde{p}_O^C$ and $\tilde{p}_{O32}^C$, then take derivative,

$$\frac{d^2\left(\tilde{p}_O^C - \tilde{p}_{O32}^C\right)}{df^2} = -\frac{l^2}{2\left(-l\left(f - s_F + 1\right)\right)^{3/2}} < 0 .$$

$$\frac{d\left(\tilde{p}_O^C - \tilde{p}_{O32}^C\right)}{df} = 1 - \frac{l}{\sqrt{-l\left(f - s_F + 1\right)}} .$$

When $f = \widehat{f}_{F3}$

$$\tilde{p}_O^C - \tilde{p}_{O32}^C = -l + 1 > 0 .$$

When $f = \widehat{f}_{F2}$

$$\frac{d\left(\tilde{p}_O^C - \tilde{p}_{O32}^C\right)}{df} = \frac{\sqrt{2}}{1+\sqrt{2}} > 0\,.$$

So $\tilde{p}_O^C - \tilde{p}_{O32}^C = -l - 2 + s_F < 0$.

Hence F5-2 cannot be feasible.

Case F5-3: $f > \hat{f}_{F2}$ and $p_O > \hat{p}_{O23}$.

Compare $\tilde{p}_O^C$ and $\tilde{p}_{O23}^C$, then take derivative,

$$\frac{d\left(\tilde{p}_O^C - \tilde{p}_{O23}^C\right)}{df} = 1 > 0\,.$$

When $s_F = l + 2$

$$\tilde{p}_O^C - \tilde{p}_{O23}^C = 0\,.$$

When $f = \widehat{f}_{F2}$

$$\frac{d\left(\tilde{p}_O^C - \tilde{p}_{O32}^C\right)}{df} = \frac{\sqrt{2}}{1+\sqrt{2}} > 0\,.$$

So $\tilde{p}_O^C - \tilde{p}_{O32}^C = -l - 2 + s_F < 0$.

Hence F5-3 cannot be feasible.

There is no equilibrium in this case.

- **Case F5 (F) + O3 (E-S-F):**

  Solve $\left\{p_F = p_{\mathrm{O}} - l, p_{\mathrm{O}} = \frac{p_F}{2}\right\}$, which gives $\left\{\tilde{p}_F^C = -2l, \tilde{p}_O^C = -l\right\}$.

  Hence there is no equilibrium in this case as price is negative.

- **Case F5 (F) + O4 (E-S-F):**

  There is no solution to $\left\{p_F = p_{\mathrm{O}} - l, p_{\mathrm{O}} = p_F - 1\right\}$, indicating that there is no intersection.

  Hence there is no equilibrium in this case.

In summary, we define the following: $l_{A1} = 1 - \frac{\sqrt{2}}{2}$, $s_{FAl} = l + 2$, $s_{FA2} = 3\sqrt{2}l - \sqrt{2} + 4l - 1$ and update the assumption set as:

$$\left\{0 < l < l_{A1} \text{ and } 0 < s_F < s_{FA1}\right\} + \left\{l_{A1} < l < \frac{1}{3} \text{ and } s_{FA2} < s_F < s_{FA1}\right\}$$

To describe the E-S-F equilibrium we define the following:

$$l_{11} = -\frac{1}{2} + \frac{\sqrt{13}}{6}, l_{12} = \frac{\sqrt{2}}{3} - \frac{1}{3}, l_{13} = \frac{8\sqrt{2}}{17} - \frac{25}{51}, l_{14} = \frac{2\sqrt{2}}{3} - \frac{2}{3},$$
$$s_{FI11} = 1 + \frac{1}{9l}, s_{FI12} = -2\sqrt{2}l + \frac{2\sqrt{2}}{3} + \frac{5}{3} - 3l, s_{FI13} = -4\sqrt{2}l - 6l + \frac{7}{3} + \frac{4\sqrt{2}}{3},$$
$$f_{11} = \frac{1}{9l}, f_{12} = s_F - 1 - \frac{1}{9l}, f_{13} = -2\sqrt{2}l - 3l + \frac{2}{3} + \frac{2\sqrt{2}}{3}, f_{14} = s_F + 2\sqrt{2}l - \frac{2\sqrt{2}}{3} - \frac{5}{3} + 3l.$$

Then the feasible range for E-S-F equilibrium writes:

$$\{0 < l < l_{11} \text{ and } 0 < s_F < s_{FA1}\} + \{l_{11} < l < l_{12} \text{ and } 0 < s_F < s_{FI11}\} : 0 < f < f_{11},$$
$$\{l_{11} < l < l_{12} \text{ and } s_{FI11} < s_F < s_{FA1}\} : f_{12} < f < f_{11},$$
$$\{l_{12} < l < l_{14} \text{ and } 0 < s_F < s_{FI12}\} : 0 < f < f_{13},$$
$$\{l_{12} < l < l_{13} \text{ and } s_{FI12} < s_F < s_{FA1}\} + \{l_{13} < l < l_{14} \text{ and } s_{FI12} < s_F < s_{FI13}\} : f_{14} < f < f_{13}.$$

To describe the E-F equilibrium we define the following:

$$l_{21} = -\frac{5}{7} + \frac{4\sqrt{2}}{7}, l_{22} = 3\sqrt{2} - 4, s_{F21} = \frac{11}{7} - \frac{18\sqrt{2}l}{7} + \frac{6\sqrt{2}}{7} - \frac{26l}{7}, s_{F22} = 2 - \frac{3\sqrt{2}l}{2} - 2l,$$
$$f_{21} = \frac{s_F}{3} + \frac{1}{3} - \frac{5\sqrt{2}l}{3} - \frac{\sqrt{2}s_F}{3} + \frac{2\sqrt{2}}{3} - \frac{7l}{3}, f_{22} = \frac{5\sqrt{2}l}{3} + \frac{2s_F}{3} + \frac{\sqrt{2}s_F}{3} - \frac{2\sqrt{2}}{3} - \frac{4}{3} + \frac{7l}{3}$$

Then the feasible range for E-F equilibrium writes:

$$\left\{l_{21} < l < l_{22} \text{ and } s_{F21} < s_F < s_{FAl}\right\} + \left\{l_{22} < l < \frac{1}{3} \text{ and } s_{F22} < s_F < s_{FA1}\right\} : f_{21} < f < f_{22}$$

D. Comparison of Profits and Consumer Surplus

Under the C scenario, first, let us consider the equilibrium under E-F. Recall from section C, the equilibrium prices are:

$$\left\{p_F = -\frac{l}{3} + f + \frac{4}{3} - \frac{2s_F}{3}, p_{\mathrm{O}} = \frac{l}{3} + f + \frac{2}{3} - \frac{s_F}{3}\right\}.$$

From section B, we have the profit functions:

$$\tilde{\pi}_{F4}^{C} = \frac{1}{2}f - \frac{5}{18}s_F - \frac{2}{9}l + \frac{1}{36}s_F^2 + \frac{1}{36}l^2 - \frac{1}{18}ls_F + \frac{4}{9},$$

$$\tilde{\pi}_{O4}^{C} = \frac{\left(l + 2 - s_F\right)^2}{36}.$$

The consumer surplus is defined as,

$$\mathrm{CS}_{O1} = \frac{5}{18}s_F - \frac{7}{9}l - \frac{1}{2}f + \frac{1}{72}s_F^2 + \frac{1}{2}v + \frac{1}{72}l^2 - \frac{1}{36}ls_F - \frac{11}{18}.$$

Under E-S-F, we have the following prices, profits, and consumer surplus,

$$\left\{p_F = \frac{2}{3}, p_{\mathrm{O}} = \frac{1}{3}\right\}$$

$$\tilde{\pi}_{F3}^{C} = \frac{2}{9} + \frac{1}{2}fl - \frac{1}{2}ls_F$$

$$\tilde{\pi}_{O3}^{C} = \frac{1}{18} - \frac{fl}{2}$$

$$\mathrm{CS}_{O2} = \frac{1}{4}l^2 - \frac{11}{36} - l + \frac{1}{2}v$$

Similarly, under F, we have the following prices, profits, and consumer surplus,

$$\left\{p_F = p_{\mathrm{O}} - l, p_{\mathrm{O}} = p_{\mathrm{O}}\right\}$$

$$\tilde{\pi}_{FL2}^{C} = \frac{p_{\mathrm{O}}}{2} - \frac{l}{2}$$

$$\tilde{\pi}_{OL1}^{C} = 0$$

$$\mathrm{CS}_{O3} = -\frac{l}{2} + \frac{v}{2} - \frac{p_{\mathrm{O}}}{2}$$

Recall we have E-S-F equilibriums when:

$\{0 < l < l_{11}$ and $0 < s_F < s_{FA1}\} + \{l_{11} < l < l_{12}$ and $0 < s_F < s_{FI11}\} : 0 < f < f_{11}$,

$\{l_{11} < l < l_{12}$ and $s_{FI11} < s_F < s_{FA1}\} : f_{12} < f < f_{11}$,

$\{l_{12} < l < l_{14}$ and $0 < s_F < s_{FI12}\} : 0 < f < f_{13}$,

$\{l_{12} < l < l_{13}$ and $s_{FI12} < s_F < s_{FA1}\} + \{l_{13} < l < l_{14}$ and $s_{FI12} < s_F < s_{FI13}\} : f_{14} < f < f_{13}$.

and we have E-S-F equilibriums when:

$$\{l_{21} < l < l_{22} \text{ and } s_{F21} < s_F < s_{FAl}\} + \left\{l_{22} < l < \frac{1}{3} \text{ and } s_{F22} < s_F < s_{FA1}\right\}: f_{21} < f < f_{22}$$

Under N scenario, $0 < l < \frac{1}{3}$ and $s_O < \min(s_{R1}, s_{R2})$, we have the following prices, profits, and consumer surplus,

$$\left\{p_F = \frac{2}{3}, p_{\mathrm{O}} = \frac{1}{3}\right\}$$

$$\tilde{\pi}_F^N = \frac{2}{9},$$

$$\tilde{\pi}_O^N = \frac{1}{18} - s_O l + \frac{1}{2} s_O h_r.$$

$$\mathrm{CS}_N = -l - \frac{11}{36} + \frac{1}{2}v + l^2 - lh_r + \frac{1}{4}h_r^2$$

Now, we split the above conditions into five cases; for the rest of the comparison, the N scenario is always the benchmark.

<u>Case 1 (E-S-F)</u>: $\{0 < l < l_{11} \text{ and } 0 < s_F < s_{FAl}\}$ + $\{l_{11} < l < l_{12} \text{ and } 0 < s_F < s_{F11}\}$ + $\{l_{12} < l < l_{14} \text{ and } 0 < s_F < s_{F12}\}$.

In this case, the e-tailer becomes better off, the physical retailer becomes worse off, and the consumer surplus becomes better off.

<u>Case 2 (E-S-F)</u>: $\{l_{11} < l < l_{12} \text{ and } s_{F11} < s_F < s_{FA1}\}$.

In this case, the e-tailer becomes better off when $\left\{s_O > -\frac{9l^2 + 9l - 1}{9(-2l + h_r)} \text{ and } s_{F11} < s_F < s_{FAl}\right\}$.+ $\left\{0 < s_O < -\frac{9l^2 + 9l - 1}{9(-2l + h_r)} \text{ and } s_{F11} < s_F < -\frac{s_O h_r}{l} + 2s_O + 1 + \frac{1}{9l}\right\}$, and worse off when $\left\{0 < s_O < -\frac{9l^2 + 9l - 1}{9(-2l + h_r)} \text{ and } -\frac{s_O h_r}{l} + 2s_O + 1 + \frac{1}{9l} < s_F < s_{FAl}\right\}$.

The physical retailer becomes worse off; and the consumer surplus becomes better off.

<u>Case 3 (E-S-F)</u>: $\{l_{11} < l < l_{12} \text{ and } s_{F11} < s_F < s_{FA1}\}$.

In this case, for $\{l_{12} < l < l_{13} \text{ and } s_{F12} < s_F < s_{FA1}\}$:

the e-tailer: becomes better off when $\left\{s_O > \frac{(2+\sqrt{2})(-12l - 6 + 5\sqrt{2})l}{6(-2l + h_r)} \text{ and } s_{F12} < s_F < s_{FA1}\right\}$ +

$\left\{0 < s_O < \frac{(2+\sqrt{2})(-12l-6+5\sqrt{2})l}{6\left(-2l+h_r\right)} \text{ and } s_{F12} < s_F < -2\sqrt{2}l + \frac{2\sqrt{2}}{3} - \frac{s_O h_r}{l} - 3l + 2s_O + \frac{5}{3}\right\}$, and worse off when $\left\{0 < s_O < \frac{(2+\sqrt{2})(-12l-6+5\sqrt{2})l}{6\left(-2l+h_r\right)} \text{ and } -2\sqrt{2}l + \frac{2\sqrt{2}}{3} - \frac{s_O h_r}{l} - 3l + 2s_O + \frac{5}{3} < s_F < s_{FAl}\right\}$.

For $\left\{l_{13} < l < l_{14} \text{ and } s_{F12} < s_F < s_{F13}\right\}$:

the e-tailer: becomes better off when $\left\{s_O > -\frac{(3+2\sqrt{2})(-3l-2+2\sqrt{2})l}{3\left(-2l+h_r\right)} \text{ and } s_{F12} < s_F < s_{F13}\right\}$.+ $\left\{0 < s_O < -\frac{(3+2\sqrt{2})(-3l-2+2\sqrt{2})l}{3\left(-2l+h_r\right)} \text{ and } s_{F12} < s_F < -2\sqrt{2}l + \frac{2\sqrt{2}}{3} - \frac{s_O h_r}{l} - 3l + 2s_O + \frac{5}{3}\right\}$, and worse off when $\left\{0 < s_O < -\frac{(3+2\sqrt{2})(-3l-2+2\sqrt{2})l}{3\left(-2l+h_r\right)} \text{ and } -2\sqrt{2}l + \frac{2\sqrt{2}}{3} - \frac{s_O h_r}{l} - 3l + 2s_O + \frac{5}{3} < s_F < s_{F13}\right\}$.

The physical retailer becomes worse off; and the consumer surplus becomes better off.

<u>Case 4 (E-F)</u>: $\left\{l_{13} < l < l_{14} \text{ and } s_{F13} < s_F < s_{FAl}\right\}$.

In this case, the e-tailer: becomes better off when $\left\{s_O > \frac{(81+56\sqrt{2})(-51l-49+45\sqrt{2})(-51l-1+3\sqrt{2})}{46818\left(-2l+h_r\right)} \text{ and } s_{F13} < s_F < l+2-\sqrt{18s_O h_r - 36 s_O l + 2}\right\}$, and worse off when $\left\{s_O > \frac{(81+56\sqrt{2})(-51l-49+45\sqrt{2})(-51l-1+3\sqrt{2})}{46818\left(-2l+h_r\right)} \text{ and } l+2-\sqrt{18s_O h_r - 36 s_O l + 2} < s_F < s_{FAl}\right\}$ + $\left\{0 < s_O < \frac{(81+56\sqrt{2})(-51l-49+45\sqrt{2})(-51l-1+3\sqrt{2})}{46818\left(-2l+h_r\right)} \text{ and } s_{F13} < s_F < s_{FAl}\right\}$.

The physical retailer becomes better off when

$$\left\{\begin{array}{l} l_3 < l < \frac{(168\sqrt{2}-243)\sqrt{591+406\sqrt{2}}}{867} - \frac{161\sqrt{2}}{289} + \frac{1096}{867} \text{ and } s_{F13} < s_F < s_{FAl} \\ \text{and } \frac{4}{9}l + \frac{1}{9}ls_F - \frac{1}{18}s_F^2 + \frac{5}{9}s_F - \frac{1}{18}l^2 - \frac{4}{9} < f < f_{22} \end{array}\right\} +$$

$$\left\{\begin{array}{l} \frac{(168\sqrt{2}-243)\sqrt{591+406\sqrt{2}}}{867} - \frac{161\sqrt{2}}{289} + \frac{1096}{867} < l < l_{14} \text{ and } -3\sqrt{2}+l-1 \\ +\sqrt{(1+\sqrt{2})(-36l+1+17\sqrt{2})} < s_F < s_{FAl} \text{ and } f_{2l} < f < \frac{4}{9}l + \frac{1}{9}ls_F - \frac{1}{18}s_F^2 + \frac{5}{9}s_F - \frac{1}{18}l^2 - \frac{4}{9} \end{array}\right\}$$, and worse

off when $\left\{\begin{array}{l}\dfrac{(168\sqrt{2}-243)\sqrt{591+406\sqrt{2}}}{867}-\dfrac{161\sqrt{2}}{289}+\dfrac{1096}{867}<l<l_{14} \\ \text{and } s_{F13}<s_F<-3\sqrt{2}+l-1+\sqrt{(1+\sqrt{2})(-36l+1+17\sqrt{2})} \text{ and } f_{21}<f<f_{22}\end{array}\right\}$ +

$\left\{\begin{array}{l}l_3<l<\dfrac{(168\sqrt{2}-243)\sqrt{591+406\sqrt{2}}}{867}-\dfrac{161\sqrt{2}}{289}+\dfrac{1096}{867} \text{ and } s_{F13}<s_F<s_{FA1} \\ \text{and } f_{21}<f<\dfrac{4}{9}l+\dfrac{1}{9}ls_F-\dfrac{1}{18}s_F^2+\dfrac{5}{9}s_F-\dfrac{1}{18}l^2-\dfrac{4}{9}\end{array}\right\}$ +

$\left\{\begin{array}{l}\dfrac{(168\sqrt{2}-243)\sqrt{591+406\sqrt{2}}}{867}-\dfrac{161\sqrt{2}}{289}+\dfrac{1096}{867}<l<l_{14} \text{ and } -3\sqrt{2}+l-1+ \\ \sqrt{(1+\sqrt{2})(-36l+1+17\sqrt{2})}<s_F<s_{FA1} \text{ and } f_{2l}<f<\dfrac{4}{9}l+\dfrac{1}{9}ls_F-\dfrac{1}{18}s_F^2+\dfrac{5}{9}s_F-\dfrac{1}{18}l^2-\dfrac{4}{9}\end{array}\right\}$.

The consumer surplus becomes better off when

$\left\{s_{F13}<s_F<s_{FAl} \text{ and } f_{21}<f<\dfrac{4}{9}l-\dfrac{1}{18}ls_F+\dfrac{1}{36}s_F^2+\dfrac{5}{9}s_F-\dfrac{11}{18}-\dfrac{71}{36}l^2+2lh_r-\dfrac{1}{2}h_r^2\right\}$, and worse off when

$\left\{s_{Fl3}<s_F<s_{FAl} \text{ and } \dfrac{4}{9}l-\dfrac{1}{18}ls_F+\dfrac{1}{36}s_F^2+\dfrac{5}{9}s_F-\dfrac{11}{18}-\dfrac{71}{36}l^2+2lh_r-\dfrac{1}{2}h_r^2<f<f_{22}\right\}$.

Case 5 (E-F): $\left\{l_{14}<l<\dfrac{1}{3} \text{ and } s_{F22}<s_F<s_{FA1}\right\}$.

In this case, the e-tailer: becomes better off when $s_{F22}<s_F<l+2-\sqrt{18s_Oh_r-36s_Ol+2}$, and worse off when $l+2-\sqrt{18s_Oh_r-36s_Ol+2}<s_F<s_{FA1}$.

The physical retailer becomes better off when

$\left\{-3\sqrt{2}+l-1+\sqrt{(1+\sqrt{2})(-36l+1+17\sqrt{2})}<s_F<s_{FAl} \text{ and } \dfrac{4}{9}l+\dfrac{1}{9}ls_F-\dfrac{1}{18}s_F^2+\dfrac{5}{9}s_F-\dfrac{1}{18}l^2-\dfrac{4}{9}<f<f_{22}\right\}$,

and worse off when

$\left\{-3\sqrt{2}+l-1+\sqrt{(1+\sqrt{2})(-36l+1+17\sqrt{2})}<s_F<s_{FAl} \text{ and } f_{21}<f<\dfrac{4}{9}l+\dfrac{1}{9}ls_F-\dfrac{1}{18}s_F^2+\dfrac{5}{9}s_F-\dfrac{1}{18}l^2-\dfrac{4}{9}\right\}$

$+\left\{s_{F22}<s_F<-3\sqrt{2}+l-1+\sqrt{(1+\sqrt{2})(-36l+1+17\sqrt{2})} \text{ and } f_{21}<f<f_{22}\right\}$.

The consumer surplus becomes better off when $\left\{s_{F22}<s_F<s_{FA1} \text{ and } f_{21}<f<\dfrac{4}{9}l-\dfrac{1}{18}ls_F+\dfrac{1}{36}s_F^2+\dfrac{5}{9}s_F-\dfrac{11}{18}-\dfrac{71}{36}l^2+2lh_r-\dfrac{1}{2}h_r^2\right\}$, and worse off when

$\left\{s_{F22}<s_F<s_{FAl} \text{ and } \dfrac{4}{9}l-\dfrac{1}{18}ls_F+\dfrac{1}{36}s_F^2+\dfrac{5}{9}s_F-\dfrac{11}{18}-\dfrac{71}{36}l^2+2lh_r-\dfrac{1}{2}h_r^2<f<f_{22}\right\}$.

Note that for Cases 1, 2, and 3, the physical retailer is always worse off. In order to identify the win-win situation, we focus on Cases 4 and 5. Next, we consolidate Case 4 and Case 5.

Define $s_{F3} = l + 2 - \sqrt{18 s_O h_r - 36 s_O l + 2}$, $s_{F4} = -3\sqrt{2} + l - 1 + \sqrt{(1+\sqrt{2})(-36l + 1 + 17\sqrt{2})}$, $f_{C1} = \frac{4}{9}l + \frac{1}{9}ls_F - \frac{1}{18}s_F^2 + \frac{5}{9}s_F - \frac{1}{18}l^2 - \frac{4}{9}$, and $f_{C2} = \frac{4}{9}l - \frac{1}{18}ls_F + \frac{1}{36}s_F^2 + \frac{5}{9}s_F - \frac{11}{18} - \frac{71}{36}l^2 + 2lh_r - \frac{1}{2}h_r^2$.

Case 4+5 (E-F): $\{l_{13} < l < l_{14} \text{ and } s_{F13} < s_F < s_{FA1}\} + \left\{l_{14} < l < \frac{1}{3} \text{ and } s_{F22} < s_F < s_{FA1}\right\}$.

In this case, the e-tailer becomes better off when $s_F < s_{F3}$, and worse off when $s_F > s_{F3}$.

The physical retailer becomes better off when $\{s_F > s_{F4} \text{ and} f > f_1\}$, and worse off when $\{s_F > s_{F4} \text{ and} f_{21} < f < f_1\} + \{s_F < s_{F4} \text{ and} f_{21} < f < f_{22}\}$.

The consumer surplus becomes better off when $\{s_{F22} < s_F < s_{FA1} \text{ and} f_{21} < f < f_2\}$, and worse off when $\{s_{F22} < s_F < s_{FAl} \text{ and} f_2 < f < f_{22}\}$.

To identify a win-win region, we need to find the conditions for $s_{F3} > s_{F4}$. Compare s_{F3}, s_{F4},

$$s_{F3} - s_{F4} = 3 - \sqrt{18 s_O h_r - 36 s_O l + 2} + 3\sqrt{2} - \sqrt{(1+\sqrt{2})(-36l + 1 + 17\sqrt{2})}$$

Take the derivative

$$\frac{d(s_{F3} - s_{F4})}{ds} = -\frac{18h_r - 36l}{2\sqrt{18h_r s_O - 36ls_O + 2}} > 0$$

Solve, and we need

$$s_O > -\frac{\sqrt{2}\sqrt{(1+\sqrt{2})(-36l+1+17\sqrt{2})} + 6\sqrt{2}l - 6\sqrt{2} + \sqrt{(1+\sqrt{2})(-36l+1+17\sqrt{2})} + 6l - 10}{3(-2l + h_r)}.$$

Hence the win-win condition is $s_{F4} < s_F < s_{F3}$.

Note that in this case, the consumer surplus becomes worse.