Online Appendix

to

Frenemies in the Retail Market: A Partnership Between a Physical Retailer and an E-tailer for Consumer Returns

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A. Proof of Lemma 1

We first derive their shopping choices by comparing U_S, U_F, and U_E. We find that $U_E > U_S$ when $h_O < \hat{h}^i_{OES} = 2l - \phi$. For no-cross-return case, we further derive $\hat{h}^N_{OES} = 2l - h_r$. We consider that $l > h_r / 2$, such that showrooming will not dominate e-Direct. For cross-return case, we get $\hat{h}^C_{OES} = l$. We find $U_S > U_F$ when $h_O < \hat{h}^i_{OSF} = p_F - p_O$ for both cross- and no-cross-return cases.

Then we separate our analysis into two cases: (i) $\hat{h}^i_{OES} \le \hat{h}^i_{OSF}$ and (ii) $\hat{h}^i_{OES} > \hat{h}^i_{OSF}$. For the case with $\hat{h}^i_{OES} \le \hat{h}^i_{OSF}$, we get $p_O \le \hat{p}^i_{O2} = p_F - 2l + \phi$, which indicates $\hat{p}^N_{O2} = p_F - 2l + h_r$ and $\hat{p}^C_{O2} = p_F - l$. Then, we find that (i) $U_E > \max\{U_S, U_F\}$ for $0 \le h_O < \hat{h}^i_{OES}$, and (ii) $U_S \ge \max\{U_E, U_F\}$ for $\hat{h}^i_{OES} \le h_O \le \hat{h}^i_{OSF}$. If we further have $\hat{h}^i_{OSF} \le 1$, i.e., $p_O \ge \hat{p}^i_{O3} = p_F - 1$, we will have $U_F > \max\{U_S, U_E\}$ for $\hat{h}^i_{OSF} < h_O \le 1$. To summarize, when $\hat{p}^i_{O3} < p_O \le \hat{p}^i_{O2}$, the consumers with $0 \le h_O < \hat{h}^i_{OES}$ will choose e-Direct, the consumers with $\hat{h}^i_{OES} \le h_O \le \hat{h}^i_{OSF}$ will choose showrooming, and the consumers with $\hat{h}^i_{OSF} < h_O \le 1$ will choose buy-offline. If $\hat{h}^i_{OSF} > 1$, i.e., $p_O < \hat{p}^i_{O3}$, none of the consumers will choose buy-offline. The consumers with $0 \le h_O < \hat{h}^i_{OES}$ will choose e-Direct, and the consumers with $\hat{h}^i_{OES} \le h_O \le 1$ will choose showrooming. We assume that $l < (1 + h_r)/2$ in order to have $\hat{p}^N_{O3} < \hat{p}^N_{O2}$, otherwise buy-offline and showrooming would not co-exist at any given p_O for no-cross-return case.

For the case with $\hat{h}^i_{OES} > \hat{h}^i_{OSF}$, which indicates $p_O > \hat{p}^i_{O2}$, there does not exist a region for $U_S \ge \max\{U_E, U_F\}$ as it requires $\hat{h}^i_{OES} \le h_O \le \hat{h}^i_{OSF}$. Hence, there is no showrooming consumer in this case. Instead, we find that $U_E > U_F$ when $h_O < \hat{h}^i_{OEF} = (p_F - p_O + 2l - \phi)/2$, which indicates

$\hat{h}_{OEF}^{N} = \left(p_F - p_O + 2l - h_r\right)/2$ and $\hat{h}_{OEF}^{C} = \left(p_F - p_O + l\right)/2$. To make sure $\hat{h}_{OEF}^{i} > 0$, we need $p_O < \hat{p}_{O1}^{i} = p_F + 2l - \phi$, more specifically, $\hat{p}_{O1}^{N} = p_F + 2l - h_r$ and $\hat{p}_{O1}^{C} = p_F + l$. It's trivial to show $\hat{p}_{O1}^{C} > \hat{p}_{O2}^{C}$. We can further verify that $\hat{p}_{O1}^{N} > \hat{p}_{O2}^{N}$ based on the assumption $h_r / 2 < l < \left(1 + h_r\right)/2$. In addition, we find that $0 < \hat{h}_{OEF}^{i} < 1$ when $\hat{p}_{O2}^{i} < p_O \le \hat{p}_{O1}^{i}$. Hence, when $\hat{p}_{O2}^{i} < p_O \le \hat{p}_{O1}^{i}$, the consumers with $0 \le h_O \le \hat{h}_{OEF}^{i}$ will choose e-Direct, and the consumers with $\hat{h}_{OEF}^{i} < h_O \le 1$ will choose buy-offline. When $p_O > \hat{p}_{O1}^{i}$, we have $\hat{h}_{OEF}^{i} \le 0$. In such a case, the consumers with $0 \le h_O \le 1$ will choose buy-offline.

B. Proof of Lemma 2

We first set up the consumer demand a , based on consumer segmentation from Lemma 1. For simplicity, we introduce the following notation: we use case A to denote Seg F (segment F) from Lemma 1, case B for Seg E-F, case C for Seg E-S-F, and case D for Seg E-S.

- Case A: When $p_O > \hat{p}_{O1}^{i}$, $a_{EA}^{i} = 0$, $a_{SA}^{i} = 0$, $a_{FA}^{i} = 1/2$;
- Case B: When $\hat{p}_{O2}^{i} < p_O \le \hat{p}_{O1}^{i}$, $a_{EB}^{i} = \hat{h}_{OEF}^{i} / 2$, $a_{SB}^{i} = 0$, $a_{FB}^{i} = \left(1 - \hat{h}_{OEF}^{i}\right)/2$;
- Case C: When $\hat{p}_{O3}^{i} < p_O \le \hat{p}_{O2}^{i}$, $a_{EC}^{i} = \hat{h}_{OES}^{i} / 2$, $a_{SC}^{i} = \left(\hat{h}_{OSF}^{i} - \hat{h}_{OES}^{i}\right)/2$, $a_{FC}^{i} = \left(1 - \hat{h}_{OSF}^{i}\right)/2$;
- Case D: When $p_O \le \hat{p}_{O3}^{i}$, $a_{ED}^{i} = \hat{h}_{OES}^{i} / 2$, $a_{SD}^{i} = \left(1 - \hat{h}_{OES}^{i}\right)/2$, $a_{FD}^{i} = 0$.

Now let's derive offline retailer's best response functions under each case.

- Case A: When $p_O > \hat{p}_{O1}^{C}$, we get $p_F < p_O - l$, the total profit function is $\pi_{FA} = \left(p_F\right)\cdot a_{FA}^{C} + \left(f - s_F\right)\cdot a_{EA}^{C} = p_F / 2$. We derive positive derivative $\frac{d\pi_{FA}}{dp_F} = \frac{1}{2}$, so the best response price for physical retailer is $p_F^{*} = \hat{p}_{F5}^{C} = p_{\mathrm{O}} - l$. Thus, the total profit for offline retailer in this case is $\pi_{FA}^{*} = \frac{p_O - l}{2}$;
- Case B: When $\hat{p}_{O2}^{C} < p_O \le \hat{p}_{O1}^{C}$, we get $p_O - l \le p_F < p_O + l$, the total profit function is $\pi_{FB} = \left(p_F\right)\cdot a_{FB}^{C} + \left(f - s_F\right)\cdot a_{EB}^{C} = p_F\left(\frac{1}{2} - \frac{l}{4} - \frac{p_F}{4} + \frac{p_{\mathrm{O}}}{4}\right) + \left(f - s_F\right)\left(\frac{l}{4} + \frac{p_F}{4} - \frac{p_{\mathrm{O}}}{4}\right)$. We solve the derivative $\frac{d\pi_{FB}}{dp_F} = 0$ and get $p_F^{*} = \hat{p}_{F4}^{C} = \left(p_O + f - s_F - l + 2\right)/2$ and

$\pi^*_{FB} = -\frac{1}{4}l + \frac{1}{4}p_O + \frac{1}{4} + \frac{1}{16}f^2 - \frac{1}{8}fs_F + \frac{1}{16}s_F^2 - \frac{1}{8}p_O l + \frac{1}{16}p_O^2 + \frac{1}{16}l^2 + \frac{1}{8}lf + \frac{1}{4}f - \frac{1}{8}p_O f - \frac{1}{8}ls_F - \frac{1}{4}s_F + \frac{1}{8}p_O s_F$.

Then we evaluate at the upper limit of p_F, $p_O + l - \hat{p}^C_{F4} = \frac{3l}{2} + \frac{p_O}{2} - 1 - \frac{f}{2} + \frac{s_F}{2}$. To make $p_O + l - \hat{p}^C_{F4} \geq 0$, we get $p_O \leq \hat{p}^C_{O13} = f - s_F - 3l + 2$. Then we evaluate at the lower limit of p_F, $\hat{p}^C_{F4} - p_O + l = 1 + \frac{l}{2} - \frac{p_O}{2} + \frac{f}{2} - \frac{s_F}{2}$. To make $\hat{p}^C_{F4} - p_O + l \geq 0$, we get $p_O \leq \hat{p}^C_{O14} = f - s_F + l + 2$. Note here, $\hat{p}^C_{O14} - \hat{p}^C_{O13} = 4l$ is positive. When $p_O < \hat{p}^C_{O13}$, solve the Lagrangian $\pi_{L1FB} = p_F\left(\frac{1}{2} - \frac{l}{4} - \frac{p_F}{4} + \frac{p_O}{4}\right) + \left(f - s_F\right)\left(\frac{l}{4} + \frac{p_F}{4} - \frac{p_O}{4}\right) + \lambda\left(l + p_O - p_F\right)$, we get the boundary solution $p^*_F = \hat{p}^C_{F3} = p_O + l$ and $\pi^*_{L1FB} = \frac{1}{2}l - \frac{1}{2}l^2 + \frac{1}{2}p_O - \frac{1}{2}p_O l + \frac{1}{2}lf - \frac{1}{2}ls_F$. When $p_O > \hat{p}^C_{O14}$, solve the Lagrangian $\pi_{L2FB} = p_F\left(\frac{1}{2} - \frac{l}{4} - \frac{p_F}{4} + \frac{p_O}{4}\right) + \left(f - s_F\right)\left(\frac{l}{4} + \frac{p_F}{4} - \frac{p_O}{4}\right) + \lambda\left(p_F - p_O + l\right)$, we get the boundary solution $p^*_F = \hat{p}^C_{F5} = p_O - l$ and $\pi^*_{L2FB} = \frac{p_O - l}{2}$;

- Case C: When $\hat{p}^C_{O3} < p_O \leq \hat{p}^C_{O2}$, we get $p_O + l \leq p_F < p_O + 1$, the total profit function is $\pi_{FC} = p_F\left(\frac{1}{2} - \frac{p_F}{2} + \frac{p_O}{2}\right) + \frac{\left(f - s_F\right)l}{2}$. We derive negative second order derivative $\frac{d^2\pi_{FC}}{dp_F^2} = -1$, so we get $p_F = \hat{p}^C_{F2} = \left(p_O + 1\right)/2$ such that $\frac{d\pi_{FC}}{dp_F} = 0$. The total profit in this case is $\pi^*_{FC} = \frac{1}{8} + \frac{1}{4}p_O + \frac{1}{8}p_O^2 + \frac{1}{2}lf - \frac{1}{2}ls_F$. To reach this optimal price and profit, we need to have $p_O + l \leq \hat{p}^C_{F2} < p_O + 1$. For the upper limit, $p_O + 1 - \hat{p}^C_{F2} = \left(p_O + 1\right)/2 > 0$ when $p_O > \hat{p}^C_{O11} = -1$. For the lower limit $\hat{p}^C_{F2} - p_O - l = \frac{1}{2} - \frac{p_O}{2} - l > 0$ when $p_O < \hat{p}^C_{O12} = 1 - 2l$. Notice that $\hat{p}^C_{O12} - \hat{p}^C_{O11} = 2\left(1 - l\right) > 0$, so we have $\hat{p}^C_{O11} < p_O < \hat{p}^C_{O12}$. Next we derive the boundary solution when $p_O < \hat{p}^C_{O11}$. We solve the Lagrangian

$\pi_{L1FC} = p_F\left(\frac{1}{2}-\frac{p_F}{2}+\frac{p_{\text{O}}}{2}\right)+\frac{(f-s_F)l}{2}+\lambda\left(1+p_{\text{O}}-p_F\right)$, and get the boundary solution $p_F^* = \hat{p}_{F1}^C = p_{\text{O}}+1$ and $\pi_{L1FC}^* = \frac{(f-s_F)l}{2}$. Then when $p_O > \hat{p}_{O12}^C$, we solve the Lagrangian $\pi_{L2FC} = p_F\left(\frac{1}{2}-\frac{p_F}{2}+\frac{p_{\text{O}}}{2}\right)+\frac{(f-s_F)l}{2}+\lambda\left(p_F-p_{\text{O}}-l\right)$, and get the boundary solution $p_F^* = \hat{p}_{F3}^C = p_{\text{O}}+l$ and $\pi_{L2FC}^* = \frac{1}{2}l-\frac{1}{2}l^2+\frac{1}{2}p_{\text{O}}-\frac{1}{2}p_{\text{O}}l+\frac{1}{2}lf-\frac{1}{2}ls_F$;

- Case D: When $p_O \le \hat{p}_{O3}^C$, we get $p_F > p_O+1$, the total profit function is $\pi_{FD} = (p_F)\cdot a_{FD}^C = 0$.

  Hence, we have no best response function for this case.

Next, we summarize the offline retailer's overall best response function by consolidating their best response from above.

- Case A: $p_F^* = \hat{p}_{F5}^C = p_{\text{O}}-l$ and the corresponding total profit is π_{FA}^*;
- Case B: When $p_O < \hat{p}_{O13}^C$, the boundary solution is $p_F^* = \hat{p}_{F3}^C = p_{\text{O}}+l$ and the corresponding total profit is π_{L1FB}^*.

  When $\hat{p}_{O13}^C < p_O < \hat{p}_{O14}^C$, the interior solution is $p_F^* = \hat{p}_{F4}^C = (p_O+f-s_F-l+2)/2$ and the corresponding total profit is π_{FB}^*.

  When $p_O > \hat{p}_{O14}^C$, the boundary solution is $p_F^* = \hat{p}_{F5}^C = p_{\text{O}}-l$ and the corresponding total profit is π_{L2FB}^*;
- Case C: When $p_O < \hat{p}_{O11}^C$, the boundary solution is $p_F^* = \hat{p}_{F1}^C = p_{\text{O}}+1$ and the corresponding total profit is π_{L1FC}^*.

  When $\hat{p}_{O11}^C < p_O < \hat{p}_{O12}^C$, the interior solution is $p_F^* = \hat{p}_{F2}^C = (p_O+1)/2$ and the corresponding total profit is π_{FC}^*.

  When $p_O > \hat{p}_{O12}^C$, the boundary solution is $p_F^* = \hat{p}_{F3}^C = p_{\text{O}}+l$ and the corresponding total profit is π_{L2FC}^*.

From the summary, we find $\pi_{FA}^* = \pi_{L2FB}^*$, so π_A^* is dominated. We also notice that $\pi_{L1FB}^* = \pi_{L2FC}^*$. Hence, we compare the two boundaries $\hat{p}_{O13}^C$ and $\hat{p}_{O12}^C$, and we get $\hat{p}_{O13}^C - \hat{p}_{O12}^C = -l+1+f-s_F$. We derive $\hat{p}_{O13}^C > \hat{p}_{O12}^C$ when $f > s_F+l-1$. Therefore, we have:

- Case F1: $f > \hat{f}_{F1} = s_F + l - 1$

  When $p_O < \hat{p}^C_{O11}$, $p^*_F = \hat{p}^C_{F1}$ and the total profit is π^*_{L1FC}.

  When $\hat{p}^C_{O11} < p_O < \hat{p}^C_{O12}$, $p^*_F = \hat{p}^C_{F2}$ and the total profit is π^*_{FC}.

  When $\hat{p}^C_{O12} < p_O < \hat{p}^C_{O13}$, $p^*_F = \hat{p}^C_{F3}$ and the total profit is π^*_{L1FB}.

  When $\hat{p}^C_{O13} < p_O < \hat{p}^C_{O14}$, $p^*_F = \hat{p}^C_{F4}$ and the total profit is π^*_{FB}.

  When $p_O > \hat{p}^C_{O14}$, $p^*_F = \hat{p}^C_{F5}$ and the total profit is π^*_{L2FB};

When $f < s_F + l - 1$, i.e., $\hat{p}^C_{O13} < \hat{p}^C_{O12}$, we need to compare π^*_{FB} and π^*_{FC}. Hence, we get $\pi^*_{FC} - \pi^*_{FB} = -\frac{1}{8} + \frac{1}{16} p_O^2 + \frac{3}{8} lf - \frac{3}{8} ls_F + \frac{1}{4} l - \frac{1}{16} f^2 + \frac{1}{8} fs_F - \frac{1}{16} s_F^2 + \frac{1}{8} p_O l - \frac{1}{16} l^2 - \frac{1}{4} f + \frac{1}{8} p_O f + \frac{1}{4} s_F - \frac{1}{8} p_O s_F$.
We derive positive second order derivative $\frac{d^2\left(\pi^*_{FC} - \pi^*_{FB}\right)}{dp_O^2} = \frac{1}{8}$. Then we evaluate $\pi^*_{FC} - \pi^*_{FB}$ when $p_O = \hat{p}^C_{O12}$, and we get $\pi^*_{FC} - \pi^*_{FB} = -\frac{\left(-l + 1 + f - s_F\right)^2}{16} < 0$. We evaluate $\pi^*_{FC} - \pi^*_{FB}$ when $p_O = \hat{p}^C_{O13}$, and we get $\pi^*_{FC} - \pi^*_{FB} = \frac{\left(-l + 1 + f - s_F\right)^2}{8} > 0$. After solving $\pi^*_{FC} - \pi^*_{FB} = 0$, we get two roots $p_{OA} = -\sqrt{2} f + \sqrt{2} l + \sqrt{2} s_F - \sqrt{2} - f - l + s_F$ and $p_{OB} = \sqrt{2} f - \sqrt{2} l - \sqrt{2} s_F + \sqrt{2} - f - l + s_F$. Then to compare p_{OA} and p_{OB}, we take the difference $p_{OA} - p_{OB} = -2\sqrt{2}\left(-l + 1 + f - s_F\right)$. When $f = s_F + l - 1$, we have $p_{OA} - p_{OB} = 0$. Since $\frac{d\left(p_{OA} - p_{OB}\right)}{df} = -2\sqrt{2} < 0$ and $f < s_F + l - 1$, we have $p_{OA} - p_{OB} > 0$. Therefore, the smaller root p_{OB} is inside the range and we get $\hat{p}^C_{O22} = \sqrt{2} f - \sqrt{2} l - \sqrt{2} s_F + \sqrt{2} - f - l + s_F$. Since $\frac{d\hat{p}^C_{O22}}{df} = \sqrt{2} - 1 > 0$, $\hat{p}^C_{O22}$ decrease as f decreases. Next, we will compare $\hat{p}^C_{O22}$ with $\hat{p}^C_{O11}$ and $\hat{p}^C_{O14}$. First, we get $\frac{d\hat{p}^C_{O11}}{df} = 0$ and $\frac{d\hat{p}^C_{O14}}{df} = 1$. Given $\frac{d\hat{p}^C_{O14}}{df} > \frac{d\hat{p}^C_{O22}}{df} > \frac{d\hat{p}^C_{O11}}{df}$, $\hat{p}^C_{O22}$ have a chance to intersect with $\hat{p}^C_{O11}$ and $\hat{p}^C_{O14}$. Second, let $\hat{p}^C_{O22} = \hat{p}^C_{O11}$, so we have $f_{11} = 3l + s_F - 3 + 2\sqrt{2} l - 2\sqrt{2}$. Let $\hat{p}^C_{O22} = \hat{p}^C_{O14}$, so we have $f_{14} = \hat{f}_{F2} = s_F - (3 + 2\sqrt{2}) l - 1$. Then, we compare f_{11} and f_{14}, we get

$f_{14}-f_{11}=2\left(3+2\sqrt{2}\right)\left(-l-1+\sqrt{2}\right)$. Note that $f_{14}-f_{11}>0$ when $0<l<\frac{1}{3}$. Hence, when f decreases, $\hat{p}_{O22}^{C}$ will reach $\hat{p}_{O23}^{C}=\hat{p}_{O14}^{C}$ first. Therefore, to summarize, we have:

- Case F2: $\hat{f}_{F2}<f<\hat{f}_{F1}$

  When $p_O\le\hat{p}_{O21}^{C}=\hat{p}_{O11}^{C}$, $p_F^{*}=\hat{p}_{F1}^{C}$ and the total profit is π_{L1FC}^{*}.

  When $\hat{p}_{O21}^{C}<p_O\le\hat{p}_{O22}^{C}$, $p_F^{*}=\hat{p}_{F2}^{C}$ and the total profit is π_{FC}^{*}.

  When $\hat{p}_{O22}^{C}<p_O\le\hat{p}_{O23}^{C}$, $p_F^{*}=\hat{p}_{F4}^{C}$ and the total profit is π_{FB}^{*}.

  When $p_O>\hat{p}_{O23}^{C}$, $p_F^{*}=\hat{p}_{F5}^{C}$ and the total profit is π_{L2FB}^{*};

When $f<\hat{f}_{F2}$, we have $\hat{p}_{O22}^{C}>\hat{p}_{O23}^{C}$, so we need to compare π_{FC}^{*} and π_{L2FB}^{*}. We derive $\pi_{FC}^{*}-\pi_{L2FB}^{*}=\frac{1}{8}-\frac{1}{4}p_{\mathrm{O}}+\frac{1}{8}p_{\mathrm{O}}^{2}+\frac{1}{2}lf-\frac{1}{2}ls_F+\frac{1}{2}l$ and the second order derivative $\frac{d^2\left(\pi_{FC}^{*}-\pi_{L2FB}^{*}\right)}{dp_O^2}=\frac{1}{4}$ is positive. We first evaluate $\pi_{FC}^{*}-\pi_{L2FB}^{*}$ when $p_O=\hat{p}_{O11}^{C}$, and get $\pi_{FC}^{*}-\pi_{L2FB}^{*}=\frac{1}{2}+\frac{1}{2}lf-\frac{1}{2}ls_F+\frac{1}{2}l$. Then we get $\frac{d\left(\pi_{FC}^{*}-\pi_{L2FB}^{*}\right)}{df}=\frac{l}{2}>0$. When $f=\hat{f}_{F2}$, we have $\pi_{FC}^{*}-\pi_{L2FB}^{*}=\frac{(2\sqrt{2}+3)(-l-1+\sqrt{2})(l-1+\sqrt{2})}{2}>0$, assuming $0<l<\frac{1}{3}$. Let $\pi_{FC}^{*}-\pi_{L2FB}^{*}=0$, we have $f=\hat{f}_{F3}=s_F-1-\frac{1}{l}$. Hence when $\hat{f}_{F3}<f<\hat{f}_{F2}$, we have $\frac{1}{2}+\frac{1}{2}lf-\frac{1}{2}ls_F+\frac{1}{2}l>0$. Then we evaluate $\frac{d\left(\pi_{FC}^{*}-\pi_{L2FB}^{*}\right)}{dp_O}=\frac{p_O}{4}-\frac{1}{4}$ when $p_O=\hat{p}_{O11}^{C}$, and get $\frac{d\left(\pi_{FC}^{*}-\pi_{L2FB}^{*}\right)}{dp_O}=-\frac{1}{2}<0$. Next, we derive the upper boundary of p_O by solving $\pi_{FC}^{*}-\pi_{L2FB}^{*}=0$. We get two roots $p_{OA}=1+2\sqrt{-l\left(f-s_F+1\right)}$ and $p_{OB}=1-2\sqrt{-l\left(f-s_F+1\right)}$. Then we compare p_{OA} and p_{OB}, and get $p_{OA}-p_{OB}=4\sqrt{-l\left(f-s_F+1\right)}>0$. So we pick up the smaller root and have $\hat{p}_{O32}^{C}=p_{OB}=1-2\sqrt{\left(-f+s_F-1\right)l}$. To evaluate $\hat{p}_{O32}^{C}$, we first have $\frac{d\hat{p}_{O32}^{C}}{df}=\frac{l}{\sqrt{-lf+ls_F-l}}>0$ and $\frac{d\hat{p}_{O11}^{C}}{df}=0$. Then we solve $\hat{p}_{O32}^{C}=\hat{p}_{O11}^{C}$ and get $f=\hat{f}_{F3}=s_F-1-\frac{1}{l}$. Hence, we have $\hat{p}_{O11}^{C}<\hat{p}_{O32}^{C}$. To summarize the case, we have:

- Case F3: $\hat{f}_{F3} < f < \hat{f}_{F2}$

  When $p_O \le \hat{p}_{O31}^C = \hat{p}_{O11}^C$, $p_F^* = \hat{p}_{F1}^C$ and the total profit is π_{L1FC}^*.

  When $\hat{p}_{O31}^C < p_O \le \hat{p}_{O32}^C$, $p_F^* = \hat{p}_{F2}^C$ and the total profit is π_{FC}^*.

  When $p_O > \hat{p}_{O32}^C$, $p_F^* = \hat{p}_{F5}^C$ and the total profit is π_{L2FB}^*;

When $f < \hat{f}_{F3}$, we have $\hat{p}_{O11}^C > \hat{p}_{O32}^C$, so we need to compare π_{L1FC}^* and π_{L2FB}^*. We derive $\pi_{L1FC}^* - \pi_{L2FB}^* = \frac{(f - s_F)l}{2} - \frac{p_O}{2} + \frac{l}{2}$ and after solving $\pi_{L1FC}^* - \pi_{L2FB}^* = 0$, we have $p_O = \hat{p}_{O41}^C = (f - s_F + 1)l$. To summarize, we have:

- Case F4: $f \le \hat{f}_{F3}$

  When $p_O \le \hat{p}_{O41}^C$, $p_F^* = \hat{p}_{F1}^C$ and the total profit is π_{L1FC}^*.

  When $p_O > \hat{p}_{O41}^C$, $p_F^* = \hat{p}_{F5}^C$ and the total profit is π_{L2FB}^*.

Now let's derive e-retailer's best response functions p_O^* to the offline retailer's choice of offline price under each case.

- Case A: When $p_O > \hat{p}_{O1}^C$, we get $p_O > p_F + l$, the total profit function is $\pi_{OA} = p_O \cdot \left(a_{EA}^C + a_{SA}^C\right) - f \cdot a_{EA}^C = 0$. Hence, there is no best response function in this case.

- Case B: When $\hat{p}_{O2}^C < p_O \le \hat{p}_{O1}^C$, we get $p_F - l < p_O \le p_F + l$, the total profit function is $\pi_{OB} = p_O \cdot \left(a_{EB}^C + a_{SB}^C\right) - f \cdot a_{EB}^C = p_O \left(\frac{l}{4} + \frac{p_F}{4} - \frac{p_O}{4}\right) - f\left(\frac{l}{4} + \frac{p_F}{4} - \frac{p_O}{4}\right)$ and we derive the second order derivative $\frac{d^2\pi_{OB}}{dp_O^2} = -\frac{1}{2} < 0$. Then we solve $\frac{d\pi_{OB}}{dp_O} = 0$ and get $p_O^* = \hat{p}_{O2}^C = \left(p_F + f + l\right)/2$ and $\pi_{OB}^* = \frac{\left(-l - p_F + f\right)^2}{16}$. Note that we have the condition $p_F - l < p_O \le p_F + l$, so we first evaluate the lower boundary $p_O - \left(p_F - l\right)$. When $p_O = \left(p_F + f + l\right)/2$, we get $p_O - \left(p_F - l\right) = \frac{3l}{2} - \frac{p_F}{2} + \frac{f}{2}$. We derive negative derivative $\frac{d\left(\frac{3l}{2} - \frac{p_F}{2} + \frac{f}{2}\right)}{dp_F} = -\frac{1}{2}$ and get $p_F = 3l + f$ when $\frac{3l}{2} - \frac{p_F}{2} + \frac{f}{2} = 0$. Hence, we need to have

$p_F < 3l + f$. Then we evaluate the upper boundary $p_F + l - p_O$. When $p_O = \left(p_F + f + l\right)/2$, we get $p_F + l - p_O = \frac{l}{2} + \frac{p_F}{2} - \frac{f}{2}$. We derive positive derivative $\frac{d\left(\frac{l}{2} + \frac{p_F}{2} - \frac{f}{2}\right)}{dp_F} = \frac{1}{2}$ and get $p_F = \hat{p}_{F11} = f - l$ when $\frac{l}{2} + \frac{p_F}{2} - \frac{f}{2} = 0$. Hence, we need to have $p_F > f - l$. Then, we check the compatibility and have $\left(3l + f\right) - \left(f - l\right) = 4l > 0$. So we need to satisfy the condition $f - l < p_F < 3l + f$ in this case. When $p_F < f - l$, solve the Lagrangian $\pi_{L1OB} = p_O\left(\frac{l}{4} + \frac{p_F}{4} - \frac{p_O}{4}\right) - f\left(\frac{l}{4} + \frac{p_F}{4} - \frac{p_O}{4}\right) + \lambda\left(p_F - p_O + l\right)$, we get the boundary solution $p_O^* = \hat{p}_{O1}^C = p_F + l$ and $\pi_{L1OB}^* = 0$. When $p_F > 3l + f$, we get the boundary solution $p_O^* = p_F - l$ and $\pi_{L2OB}^* = -\frac{l\left(l - p_F + f\right)}{2}$.

- Case C: When $\hat{p}_{O3}^C < p_O \le \hat{p}_{O2}^C$, we get $p_F - 1 \le p_O < p_F - l$, the total profit function is $\pi_{OC} = p_O \cdot \left(a_{EC}^C + a_{SC}^C\right) - f \cdot a_{EC}^C = \frac{p_O\left(p_F - p_O\right)}{2} - \frac{lf}{2}$ and we derive the second order derivative $\frac{d^2\pi_{OC}}{dp_O^2} = -1 < 0$. Then we solve $\frac{d\pi_{OC}}{dp_O} = 0$ and get $p_O^* = \hat{p}_{O3}^C = \frac{p_F}{2}$ and $\pi_{OC}^* = \frac{p_F^2}{8} - \frac{lf}{2}$. Note that we have the condition $p_F - 1 \le p_O < p_F - l$, so we first evaluate the lower boundary $p_O - \left(p_F - 1\right)$. When $p_O = \frac{p_F}{2}$, we get $p_O - \left(p_F - 1\right) = 1 - \frac{p_F}{2}$. We derive negative derivative $\frac{d\left(1 - \frac{p_F}{2}\right)}{dp_F} = -\frac{1}{2}$ and get $p_F = \hat{p}_{F13} = 2$ when $1 - \frac{p_F}{2} = 0$. Hence we need to have $p_F < 2$. Then we evaluate the upper boundary $p_F - l - p_O$. When $p_O = \frac{p_F}{2}$, we get $p_F - l - p_O = \frac{p_F}{2} - l$. We derive positive derivative $\frac{d\left(\frac{p_F}{2} - l\right)}{dp_F} = \frac{1}{2}$ and get $p_F = 2l$ when $\frac{p_F}{2} - l = 0$. Hence, we need to have $p_F > 2l$. Then, we check the compatibility and have

$2-2l>0$ based on our assumption that $0<l<\frac{1}{3}$. So we need to satisfy the condition $2l<p_F<2$ in this case. When $p_F<2l$, solve the Lagrangian $\pi_{L1OC}=\frac{p_{\mathrm{O}}\left(p_F-p_{\mathrm{O}}\right)}{2}-\frac{lf}{2}+\lambda\left(p_F-p_{\mathrm{O}}-l\right)$, we get the boundary solution $p_O^*=p_F-l$ and $\pi_{L1OC}^*=-\frac{l\left(l-p_F+f\right)}{2}$. When $p_F>2$, solve the Lagrangian $\pi_{L2OC}=\frac{p_{\mathrm{O}}\left(p_F-p_{\mathrm{O}}\right)}{2}-\frac{lf}{2}+\lambda\left(1+p_{\mathrm{O}}-p_F\right)$, we get the boundary solution $p_O^*=\hat{p}_{O4}^C=p_F-1$ and $\pi_{L2OC}^*=-\frac{1}{2}+\frac{p_F}{2}-\frac{lf}{2}$.

- Case D: When $p_O\leq\hat{p}_{O3}^C$, we get $p_O<p_F-1$, the total profit function is $\pi_{FD}=p_{\mathrm{O}}\cdot\left(a_{ED}^C+a_{SD}^C\right)-f\cdot a_{ED}^C=\frac{p_{\mathrm{O}}}{2}-\frac{lf}{2}$. We derive positive derivative $\frac{d\pi_{FD}}{dp_O}=\frac{1}{2}$. Hence we get the boundary solution $p_O^*=p_F-1$ and $\pi_{L1OD}^*=-\frac{1}{2}+\frac{p_F}{2}-\frac{lf}{2}$.

Next, we summarize the e-retailer's overall best response function by consolidating their best response from above. First, we notice that $\pi_{L2OC}^*=\pi_{L1OD}^*$, so case D is dominated. Therefore, we have the following:

- Case B: When $p_F<f-l$, the boundary solution is $p_O^*=\hat{p}_{O1}^C=p_F+l$ and the corresponding total profit is π_{L1OB}^*.

  When $f-l<p_F<3l+f$, the interior solution is $p_O^*=\hat{p}_{O2}^C=\left(p_F+f+l\right)/2$ and the corresponding total profit is π_{OB}^*.

  When $p_F>3l+f$, the boundary solution is $p_O^*=p_F-l$ and the corresponding total profit is π_{L2OB}^*;

- Case C+D: When $p_F<2l$, the boundary solution is $p_O^*=p_F-l$ and the corresponding total profit is π_{L1OC}^*.

  When $2l<p_F<2$, the interior solution is $p_O^*=\hat{p}_{O3}^C=p_F/2$ and the corresponding total profit is π_{OC}^*.

When $p_F > 2$, the boundary solution is $p_O^* = \hat{p}_{O4}^C = p_F - 1$ and the corresponding total profit is π_{L2OC}^*.

First, we notice that $\pi_{L2OB}^* = \pi_{L1OC}^*$. Then we compare the two boundaries $3l + f$ and $2l$, and we have $3l + f - 2l = l + f > 0$ given $f > 0$. So we get $2l < 3l + f$. Then we need to discuss the position of the other two boundaries $f - l$ and 2. Since $f - l < 3l + f$, there are two possible positions for $f - l$, i.e., $f - l < 2l < 3l + f$ and $2l < f - l < 3l + f$. Therefore, we look at the two cases separately.

When $f - l < 2l$, i.e. $f < 3l$, we have $3l + f < 6l$. Since $0 < f < \frac{1}{3}$, we get $3l + f < 2$. Then we compare π_{OB}^* with π_{OC}^*, and we get $\pi_{OB}^* - \pi_{OC}^* = \frac{1}{16} f^2 + \frac{3}{8} lf - \frac{1}{8} p_F f + \frac{1}{16} l^2 + \frac{1}{8} l p_F - \frac{1}{16} p_F^2$. We derive the second order derivative $\frac{d^2\left(\pi_{OB}^* - \pi_{OC}^*\right)}{dp_F^2} = -\frac{1}{8} < 0$. Then when $p_F = 2l$, we get $\pi_{OB}^* - \pi_{OC}^* = \frac{(l+f)^2}{16} > 0$. When $p_F = 3l + f$, we get $\pi_{OB}^* - \pi_{OC}^* = -\frac{(l+f)^2}{8} < 0$. Therefore, we derive two roots $p_{FA} = \sqrt{2} f + \sqrt{2} l - f + l$ and $p_{FB} = -\sqrt{2} f - \sqrt{2} l - f + l$ by solving $\pi_{OB}^* - \pi_{OC}^* = 0$ and we keep the larger root. We have $p_{FA} - p_{FB} = 2\sqrt{2}\left(l + f\right) > 0$, so we keep $\hat{p}_{F12} = p_{FA} = \sqrt{2}(f + l) - f + l$. Therefore, to summarize, we have:

- Case B+C+D: $f < 3l$

  When $p_F \le f - l$, the boundary solution is $p_O^* = \hat{p}_{O1}^C = p_F + l$ and the corresponding total profit is π_{L1OB}^*.

  When $f - l < p_F < \hat{p}_{F12}$, the interior solution is $p_O^* = \hat{p}_{O2}^C = \left(p_F + f + l\right)/2$ and the corresponding total profit is π_{OB}^*.

  When $\hat{p}_{F12} < p_F < 2$, the interior solution is $p_O^* = \hat{p}_{O3}^C = p_F / 2$ and the corresponding total profit is π_{OC}^*.

  When $p_F > 2$, the boundary solution is $p_O^* = \hat{p}_{O4}^C = p_F - 1$ and the corresponding total profit is π_{L2OC}^*.